The Rational Numbers consist of the fractions,

including the integers, like 0 and 1.

Under the operations $+,\ -,\ \times,\ \div$ and the relations $=, >$

these numbers satisfy the axioms for an ordered field:

$$a + b = b + a, \qquad ab = ba,$$

$$(a + b) + c = a + (b + c), \qquad (ab)c = a(bc),$$

$$0 + a = a, \qquad 1 \cdot a = a,$$

$$a + (-a) = 0, \qquad a \cdot \frac{1}{a} = 1,$$

$$a(b + c) = ab + ac,$$

$$a > 0,\ b > 0 \quad \Rightarrow \quad a + b > 0, \qquad ab > 0.$$

Their decimal expansions are usually finite, like $\frac{1}{2} = 0.5$,

or, if infinite, ultimately repeat: $\frac{1}{6} = 0.16666666666\ldots.$

The Real Numbers include the rational numbers above,
along with the irrational numbers,
like the algebraic number $\sqrt{2}$,
and the transcendental number π.
These numbers satisfy all of the operations and axioms above
along with the Least Upper Bound Axiom.
This axiom makes many other operations possible,
like $\sqrt{\ }$ and other functions.
It also makes calculus possible.

Real numbers can have infinite decimal expansions
with no restrictions of ultimate repetition:

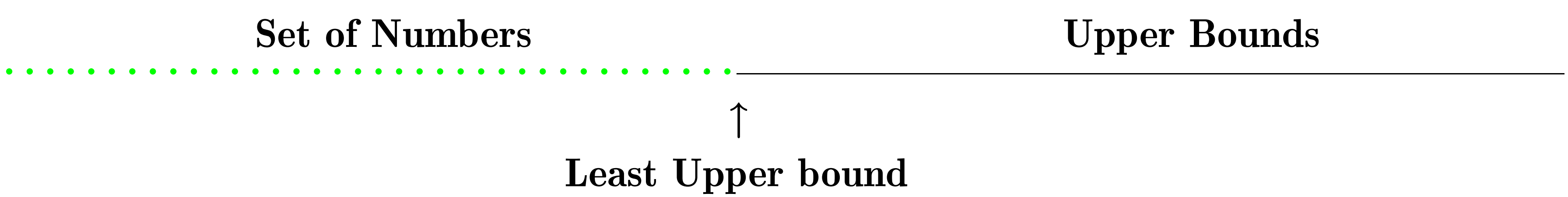
$\sqrt{3} = 1.7320508075688772935274463415058723669428....$

$\pi = 3.1415926535897932384626433832795028841971 69....$

The Least Upper Bound Axiom:

Every nonempty set of numbers having an upper bound has a least upper bound.

Every number greater than an upper bound for this set is also an upper bound for this set.
These upper bounds form a semi-infinite interval: $[L, \infty)$.



The left end of that interval is the least of the upper bounds.

This Least Upper Bound Axiom asserts that

any nonempty real number set with an upper bound

has a Least Upper Bound among the real numbers.

It may or may not be rational, or algebraic,

but it is a real number.

It may or may not belong to the set.

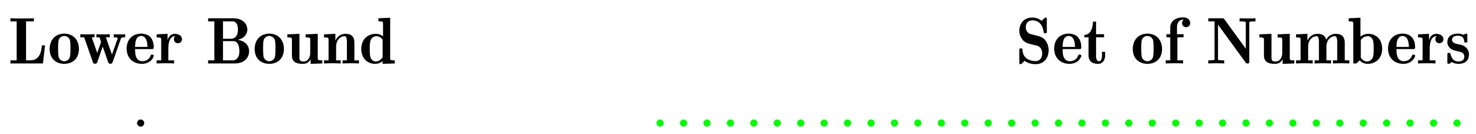
If the set has a maximum element,

that maximum element is also the set's least upper bound.

A Consequence of Least Upper Bound Axiom:

The Greatest Lower Bound Theorem:

Every nonempty set of numbers having a lower bound has a greatest lower bound.

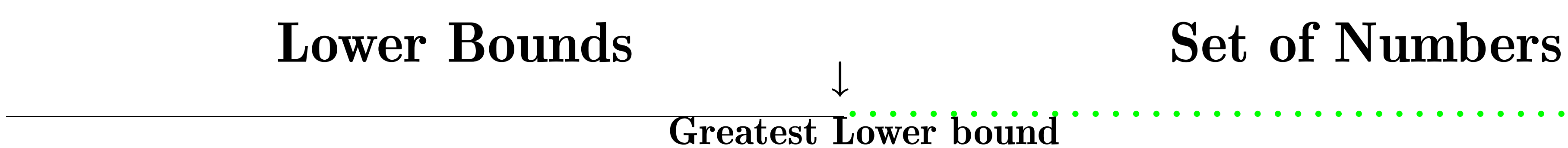


Outline of Proof:

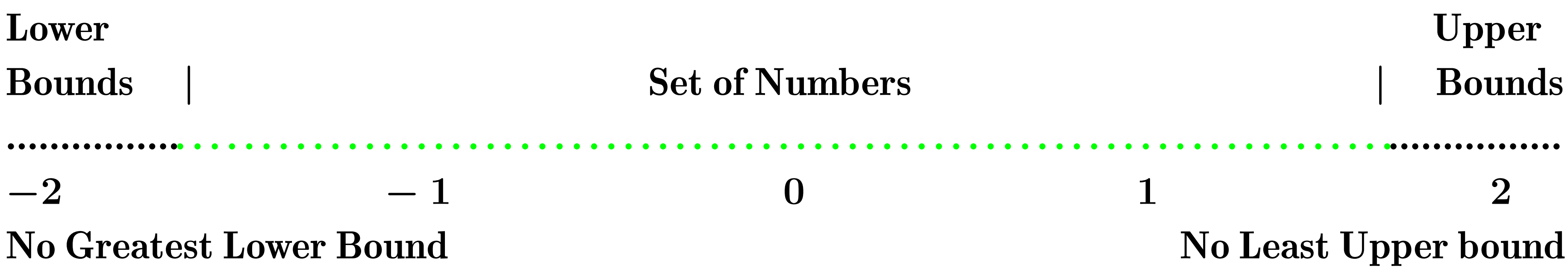
Turn things around by multiplying each set member by -1.

Apply the Least Upper Bound Axiom.

Turn things around again by multiplying everything by -1.



Example: A Set with no Rational Least Upper Bound: The Set of all Rational Numbers with Squares Less than 3, $\{\text{Rational } x | x^2 < 3\}$.



The axioms for the rational numbers

guarantee no least upper bounds.

The next slides explain why

there is no least upper bound for the set $\{x | x^2 < 3\}$

among the rational numbers.

There is no greatest lower bound among the rational numbers.

Neither is there a $\sqrt{3}$ nor a $-\sqrt{3}$ among the rational numbers.

Why The Set $\{x|x^2 < 3\}$ has no Rational Least Upper Bound:

A trichotomy: Every number x has a square x^2 which satisfies either $\boxed{x^2 = 3}$, $\boxed{x^2 < 3}$ or $\boxed{x^2 > 3}$.

To have $\boxed{x^2 = 3}$ would be impossible for any rational x:

Such an x could equal $\frac{a}{b}$, a and b having no common factor.

This would mean that $x^2 = \frac{a^2}{b^2} = 3, \quad a^2 = 3b^2$.

Then 3 would divide into a^2 and thus into a.

Let c equal $\frac{a}{3}$,

We would have: $3c = a, 9c^2 = a^2 = 3b^2$, and $3c^2 = b^2$.

Then 3 would divide into b^2 and thus into b.

A contradiction would result . $\square$

If x were negative or zero it could not be an upper bound for the positive elements in the set $\{y | y^2 < 3\}$.

Neither could any positive x satisfying $\boxed{x^2 < 3}$ be an upper bound for the set $\{y | y^2 < 3\}$,

because there would be another number $y = \dfrac{6x}{x^2 + 3}$
satisfying both $x < y$ and $y^2 < 3$:

$$\begin{aligned} x^2 < 3 \quad &\Rightarrow \quad x^2 + 3 < 6 \\ &\Rightarrow \quad 1 < \frac{6}{x^2+3} \quad \Rightarrow \quad x < \frac{6x}{x^2+3} = y, \end{aligned}$$

To show that $y^2 < 3$,
start from $x^2 < 3$, $x^2 \neq 3$, and $x^2 - 3 \neq 0$:

To show that $y^2 < 3$,

start from $x^2 < 3$, $\quad x^2 \neq 3$, and $x^2 - 3 \neq 0$:

$$\begin{aligned} 0 &< (x^2-3)^2 = x^4 - 6x^2 + 9 \\ \Rightarrow \quad 12x^2 &< x^4 + 6x^2 + 9 = (x^2+3)^2 \\ \Rightarrow \quad 36x^2 &< 3(x^2+3)^2 \\ \Rightarrow \quad \frac{36x^2}{(x^2+3)^2} &< 3. \end{aligned}$$

$$\Rightarrow y^2 = \left(\frac{6x}{x^2+3}\right)^2 < 3.$$

No x satisfying $\boxed{x^2 < 3}$

could be an upper bound for the set $\{y | y^2 < 3\}$,

or could even be a Least Upper Bound thereof.

Any positive x satisfying $\boxed{x^2 > 3}$

would be an upper bound for the set $\{y | y^2 < 3\}$,

since it would satisfy $x^2 > 3 > y^2$ and $x > y$, if $y > 0$;

but there would always be a smaller upper bound $z = \dfrac{x^2+3}{2x}$,

which would satisfy both $z < x$ and $z^2 > 3$:

$$
\begin{aligned}
x^2 > 3, \quad x > 0 \quad &\Rightarrow \quad x > \frac{3}{x} \\
&\Rightarrow \quad 2x > x + \frac{3}{x} \\
&\Rightarrow \quad x > \frac{x + \frac{3}{x}}{2} \\
&\Rightarrow \quad x > \frac{x^2+3}{2x} = z
\end{aligned}
$$

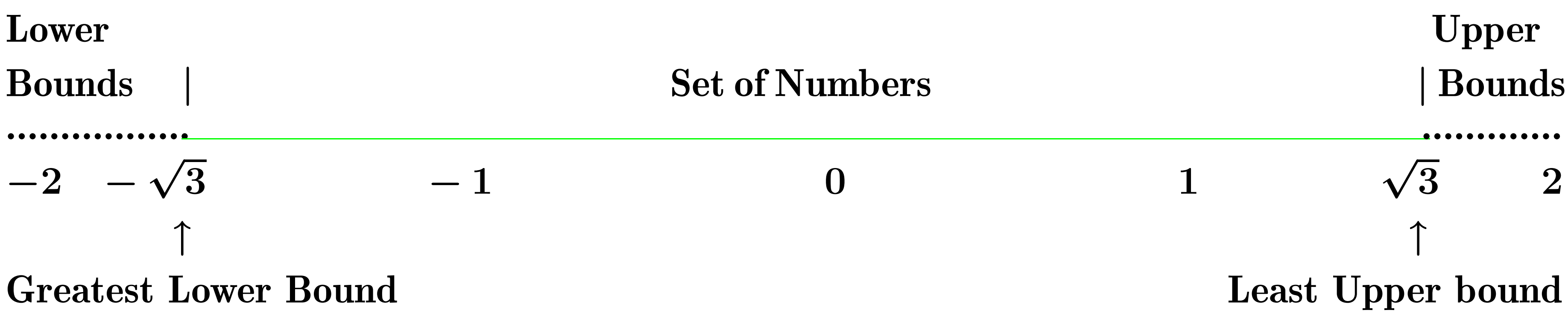
Also, $x^2 > 3$

$$
\begin{aligned}
&\Rightarrow & x^2 - 3 &> 0 \\
&\Rightarrow & (x^2 - 3)^2 &> 0 \\
&\Rightarrow & x^4 - 6x^2 + 9 &> 0 \\
&\Rightarrow & x^4 + 6x^2 + 9 &> 12x^2 \\
&\Rightarrow & (x^2 + 3)^2 &> 3(2x)^2 \\
&\Rightarrow & z^2 = \left(\frac{x^2 + 3}{2x}\right)^2 &> 3
\end{aligned}
$$

Among the rational numbers,
there is no least upper bound for the set $\{\text{Rational } x | x^2 < 3\}$,
and there is no $\sqrt{3}$.

A Happier Example:

The Set of all Real Numbers with Squares Less than 3,
$\{\text{Real } x | x^2 < 3\}$.



The Least Upper Bound Axiom for the real numbers
guarantees a least upper bound, L.
Both $L^2 < 3$ and $L^2 > 3$ would be impossible for a lub,
for the same reasons we saw on the last four slides.
Therefore, L^2 equals 3, and L is a square root of 3.

Archimedean Property: For any real number x there is at least one positive integer N such that $N > x$.

Proof by Contradiction:

If this were false,

the number x above would be an upper bound

for the positive integers,

which would also have a least upper bound L.

$L - 1$ would not be an upper bound.

There would have to be at least one integer N

satisfying $L - 1 < N \leq L$.

Adding 1, we would have $L < N + 1$,

and L would not be an upper bound at all.

Contradiction. $\square$

A consequence of the Archimedean Property:

If a nonnegative number x satisfies $0 \leq x < \frac{1}{N}$, with each positive integer N, then that x must equal 0.

Proof by Contradiction:

If this were false for any x, that x would satisfy $x > 0$.

Its reciprocal would also satisfy $\frac{1}{x} > 0$.

There would be an integer N satisfying $\frac{1}{x} < N$.

N and x would also satisfy $x > \frac{1}{N}$,

yielding a contradiction. $\square$

The Least Upper Bound Axiom yields the

Archimedean Property: For any real number x there is at least one positive integer N such that $N > x$.

Alternatively, without the Least Upper Bound Axiom, a number system could have the

Non-Archimedean Property: There is a number x such that there is no positive integer N such that $N > x$.

(Such an x would be an upper bound for the positive integers, but there would be no least upper bound for them.)

A Non-Archimedean number system would include the rational numbers and the real numbers, as we know them, along with infinite numbers and infinitesimal numbers.

There are non-Archimedean number systems in which least upper bounds exist only for non-empty sets defined by a certain type of finite "well formed" formula. (The integers are not such a set.)

Some of these "nonstandard number" systems were shown to be logically consistent during the 1900's. Calculus could be developed with these numbers but it would be very complicated for us to use here.

Leibnitz anticipated this kind of numbers in the 1600's when he intended dx and dy to be "infinitesimal" numbers. Since then, most of us have used these metaphorically, with little mathematical justification.