Factorials

0! is defined to equal 1, the multiplicative identity,
before anything else is multiplied there.

From the rule $\boxed{n! = (n-1)! \cdot n}$
1! is defined to equal $1 \cdot 1 = 1$.
2! is defined to equal $1 \cdot 2 = 2$.
3! is defined to equal $1 \cdot 2 \cdot 3 = 6$.
4! is defined to equal $1 \cdot 2 \cdot 3 \cdot 4 = 24$.
5! is defined to equal $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

For any positive integer n,
$n!$ is defined to equal $1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.

Binomial Coefficents

For positive integer n and integer k, where $0 \leq k \leq n$,

$\binom{n}{k}$ is defined as $\dfrac{n!}{k!(n-k)!}$.

$$\boxed{\binom{n}{0} = 1} \qquad \boxed{\binom{n}{n} = 1} \qquad \boxed{\binom{n}{n-k} = \binom{n}{k}}$$

$$\boxed{\binom{n}{1} = n} \qquad \boxed{\binom{n}{2} = \frac{n(n-1)}{2}}$$

Sometimes, if the integer k is negative or greater than n,

it may be useful to define $\binom{n}{k}$ to equal 0.

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
. & & . & & . & & . & & . \ \ .
\end{array}$$

Pascal's Identity: $\boxed{\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}}$

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For $0 \le k < n$:

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!\big(n-(k+1)\big)!}$$

$$= \frac{n!(k+1)}{k!(k+1)(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k-1)!(n-k)}$$

$$= \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!}$$

$$= \frac{n!(k+1+n-k)}{(k+1)!(n-k)!}$$

$$= \frac{n!(n+1)}{(k+1)!(n-k)!} = \frac{(n+1)!}{(k+1)!\Big((n+1)-(k+1)\Big)!}$$

Pascal's Identity: $\boxed{\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}}$

Verification for other k's:

For $k \leq -2$: $\binom{n}{k} + \binom{n}{k+1} = 0 + 0 = 0 = \binom{n+1}{k+1}.$

For $k = -1$: $\binom{n}{-1} + \binom{n}{0} = 0 + 1 = 1 = \binom{n+1}{0}.$

For $k = n$: $\binom{n}{n} + \binom{n}{n+1} = 1 + 0 = 1 = \binom{n+1}{n+1}.$

For $k > n$: $\binom{n}{k} + \binom{n}{k+1} = 0 + 0 = 0 = \binom{n+1}{k+1}.$

$$
\begin{aligned}
(a+b)^0 &= 1 = \binom{0}{0}a^0b^0 \\
(a+b)^1 &= a + b = \binom{1}{0}a^1b^0 + \binom{1}{1}a^0b^1 \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
&= \binom{2}{0}a^2b^0 + \binom{2}{1}a^1b^1 + \binom{2}{2}a^0b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
&= \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3 \\
(a+b)^n &= \sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k} = \sum_{\text{all } k}\binom{n}{k}a^kb^{n-k}
\end{aligned}
$$

To complete the proof for higher n's, assume

$$\begin{aligned}
(a+b)^n &= \sum_{\text{all } k} \binom{n}{k} a^k b^{n-k} \text{ for any } n : \\
(a+b)^{n+1} &= (a+b) \sum_{\text{all } k} \binom{n}{k} a^k b^{n-k} \\
&= a \sum_{\text{all } k} \binom{n}{k} a^k b^{n-k} \;+\; b \sum_{\text{all } k} \binom{n}{k} a^k b^{n-k} \\
&= \sum_{\text{all } k} \binom{n}{k} a^{k+1} b^{n-k} \;+\; \sum_{\text{all } k} \binom{n}{k} a^k b^{n-k+1} \\
&= \sum_{\text{all } k} \binom{n}{k} a^{k+1} b^{n-k} \;+\; \sum_{\text{all } k} \binom{n}{k} a^k b^{n-(k-1)} \\
&= \sum_{\text{all } j} \binom{n}{j} a^{j+1} b^{n-j} \;+\; \sum_{\text{all } j} \binom{n}{j+1} a^{j+1} b^{n-j}
\end{aligned}$$

substituting $k = j$ in ↑ and $k = j+1$, $k-1 = j$ in ↑,

Assuming

$$(a+b)^n = \sum_{\text{all } k} \binom{n}{k} a^k b^{n-k} \text{ for any } n \text{, we have :}$$

$$\begin{aligned}
(a+b)^{n+1} &= \sum_{\text{all } j} \binom{n}{j} a^{j+1} b^{n-j} \;+\; \sum_{\text{all } j} \binom{n}{j+1} a^{j+1} b^{n-j} \\
&= \sum_{\text{all } j} \left(\binom{n}{j} + \binom{n}{j+1} \right) a^{j+1} b^{n-j} \\
&= \sum_{\text{all } j} \binom{n+1}{j+1} a^{j+1} b^{n-j} \\
&= \sum_{\text{all } j} \binom{n+1}{j+1} a^{j+1} b^{(n+1)-(j+1)}
\end{aligned}$$

If we let $j+1$ equal k, we now have

$$(a+b)^{n+1} = \sum_{\text{all } k} \binom{n+1}{k} a^k b^{(n+1)-k}$$