

$$f(x)g(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} f^{(0)}(x)g^{(0)}(x)$$

$$\begin{aligned} \left(f(x)g(x) \right)' &= f'(x)g(x) + f(x)g'(x) \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} f^{(1)}(x)g^{(0)}(x) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} f^{(0)}(x)g^{(1)}(x) \end{aligned}$$

$$\begin{aligned} \left(f(x)g(x) \right)^{(2)} &= f^{(2)}(x)g(x) + 2f'(x)g'(x) + f(x)g^{(2)}(x) \\ &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} f^{(2)}(x)g^{(0)}(x) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} f^{(1)}(x)g^{(1)}(x) \\ &\quad + \begin{pmatrix} 2 \\ 2 \end{pmatrix} f^{(0)}(x)g^{(2)}(x) \end{aligned}$$

$$\begin{aligned}
\left(f(x)g(x) \right)^{(3)} &= f^{(3)}(x)g(x) + 3f^{(2)}(x)g'(x) \\
&\quad + 3f'(x)g^{(2)}(x) + f(x)g^{(3)}(x) \\
&= \binom{3}{0} f^{(3)}(x)g^{(0)}(x) + \binom{3}{1} f^{(2)}(x)g^{(1)}(x) \\
&\quad + \binom{3}{2} f^{(1)}(x)g^{(2)}(x) + \binom{3}{3} f^{(0)}(x)g^{(3)}(x)
\end{aligned}$$

Leibnitz' Formula:

$$\begin{aligned}
\left(f(x)g(x) \right)^{(n)} &= \sum_{k=0}^n \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x) \\
&= \sum_{\text{all } k} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x)
\end{aligned}$$

To complete the proof for higher n 's, assume that

$$\begin{aligned}
 \left(f(x)g(x) \right)^{(n)} &= \sum_{\text{all } k} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \text{ for any } n : \\
 \left(f(x)g(x) \right)^{(n+1)} &= \frac{d}{dx} \sum_{\text{all } k} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \\
 &= \sum_{\text{all } k} \binom{n}{k} f^{(k+1)}(x) g^{(n-k)}(x) \\
 &\quad + \sum_{\text{all } k} \binom{n}{k} f^{(k)} g^{(n-k+1)}(x) \\
 &= \sum_{\text{all } j} \binom{n}{j} f^{(j+1)}(x) g^{(n-j)}(x) \\
 &\quad + \sum_{\text{all } j} \binom{n}{j+1} f^{(j+1)}(x) g^{(n-j)}(x)
 \end{aligned}$$

substituting $k = j$ in \uparrow and, in \uparrow , $k = j + 1$, $k - 1 = j$,

Assuming

$$\begin{aligned}
 \left(f(x)g(x) \right)^{(n)} &= \sum_{\text{all } k} \binom{n}{k} f^{(k)} g^{(n-k)}(x) \text{ for any } n, \text{ we have :} \\
 \left(f(x)g(x) \right)^{(n+1)} &= \sum_{\text{all } j} \left(\binom{n}{j} + \binom{n}{j+1} \right) f^{(j+1)}(x)g^{(n-j)}(x) \\
 &= \sum_{\text{all } j} \binom{n+1}{j+1} f^{(j+1)}(x)g^{(n-j)}(x) \\
 &= \sum_{\text{all } j} \binom{n+1}{j+1} f^{(j+1)}(x)g^{((n+1)-(j+1))}(x)
 \end{aligned}$$

If we let $j + 1$ equal k , we now have

$$\left(f(x)g(x) \right)^{(n+1)} = \sum_{k=0}^n \binom{n+1}{k} f^{(k)}(x)g^{((n+1)-k)}(x).$$