$$f(x)g(x) = \binom{0}{0} f^{(0)}(x)g^{(0)}(x)$$

$$\Big(f(x)g(x)\Big)' = f'(x)g(x) + f(x)g'(x)$$

$$= \binom{1}{0} f^{(1)}(x)g^{(0)}(x) + \binom{1}{1} f^{(0)}(x)g^{(1)}(x)$$

$$\Big(f(x)g(x)\Big)^{(2)} = f^{(2)}(x)g(x) + 2f'(x)g'(x) + f(x)g^{(2)}(x)$$

$$= \binom{2}{0} f^{(2)}(x)g^{(0)}(x) + \binom{2}{1} f^{(1)}(x)g^{(1)}(x)$$

$$+ \binom{2}{2} f^{(0)}(x)g^{(2)}(x)$$

$$\Big(f(x)g(x)\Big)^{(3)} = f^{(3)}(x)g(x) + 3f^{(2)}(x)g'(x) + 3f'(x)g^{(2)}(x) + f(x)g^{(3)}(x)$$

$$= \binom{3}{0} f^{(3)}(x)g^{(0)}(x) + \binom{3}{1} f^{(2)}(x)g^{(1)}(x) + \binom{3}{2} f^{(1)}(x)g^{(2)}(x) + \binom{3}{3} f^{(0)}(x)g^{(3)}(x)$$

Leibnitz' Formula:

$$\Big(f(x)g(x)\Big)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x) = \sum_{\text{all } k} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x)$$

To complete the proof for higher n's, assume that

$$\Big(f(x)g(x)\Big)^{(n)} = \sum_{\text{all } k} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \text{ for any } n:$$

$$\begin{aligned}
\Big(f(x)g(x)\Big)^{(n+1)} &= \frac{d}{dx} \sum_{\text{all } k} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \\
&= \sum_{\text{all } k} \binom{n}{k} f^{(k+1)}(x) g^{(n-k)}(x) \\
&\qquad + \sum_{\text{all } k} \binom{n}{k} f^{(k)} g^{(n-k+1)}(x) \\
&= \sum_{\text{all } j} \binom{n}{j} f^{(j+1)}(x) g^{(n-j)}(x) \\
&\qquad + \sum_{\text{all } j} \binom{n}{j+1} f^{(j+1)}(x) g^{(n-j)}(x)
\end{aligned}$$

substituting $k = j$ in $\uparrow$ and, in $\uparrow$, $k = j + 1,\ \ k - 1 = j,$

Assuming

$$\Big(f(x)g(x)\Big)^{(n)} = \sum_{\text{all } k} \binom{n}{k} f^{(k)} g^{(n-k)}(x) \text{ for any } n, \text{ we have :}$$

$$\begin{aligned}
\Big(f(x)g(x)\Big)^{(n+1)} &= \sum_{\text{all } j} \left(\binom{n}{j} + \binom{n}{j+1}\right) f^{(j+1)}(x) g^{(n-j)}(x) \\
&= \sum_{\text{all } j} \binom{n+1}{j+1} f^{(j+1)}(x) g^{(n-j)}(x) \\
&= \sum_{\text{all } j} \binom{n+1}{j+1} f^{(j+1)}(x) g^{((n+1)-(j+1))}(x)
\end{aligned}$$

If we let $j+1$ equal k, we now have

$$\Big(f(x)g(x)\Big)^{(n+1)} = \sum_{k=0}^{n} \binom{n+1}{k} f^{(k)}(x) g^{((n+1)-k)}(x).$$