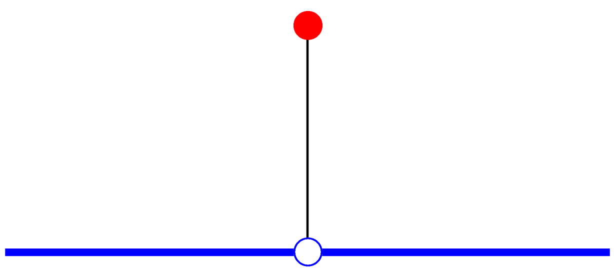
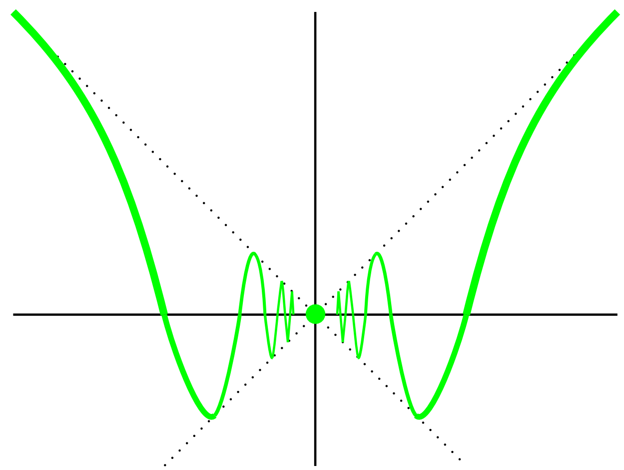
$f(g(y))$ may not have a limit, even if f and g have limits.

Separately, each of these functions have limits at all x and y:

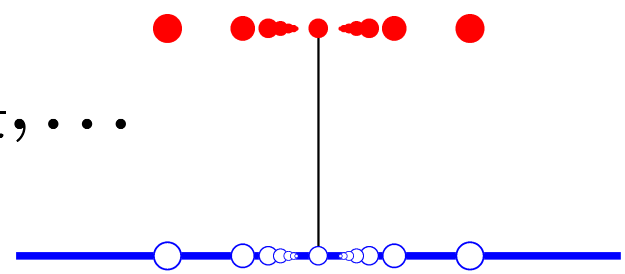
$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases} \qquad g(y) = \begin{cases} y \sin \frac{1}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0, \end{cases}$$





but $\lim\limits_{y \to 0} f(g(y))$ doesn't exist:

$$f(g(y)) = \begin{cases} 1 & \text{if } y = 0, \pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \ldots \\ 0 & \text{other } y's. \end{cases}$$



Why doesn't $f(g(x))$ have a limit? Consider the deltas and epsilons:

$$\lim_{x \to c} f(x) = L \text{ would require that}$$

For any $\epsilon_1 > 0$, there is a $\delta_1 > 0$ (depending on ϵ_1, f and c) such that $0 < |\, x - c \,| < \delta_1$ would imply $|\, f(x) - L \,| < \epsilon_1$.

If x were replaced by $g(y)$, this would become

For any $\epsilon_1 > 0$, there is a $\delta_1 > 0$ (depending on ϵ_1, f and c) such that $\boxed{0 < |\, g(y) - c \,| < \delta_1}$

would imply $|\, f(g(y)) - L \,| < \epsilon_1$,

(In particular, this would need $g(y) \neq c$.)

On the other hand, $\lim_{y \to d} g(y) = c$ would mean that

For our $\epsilon_2 = \delta_1 > 0$ there is a $\delta_2 > 0$

(depending on δ_1, g and d)

such that $0 < |\, y - d \,| < \delta_2$

would imply only that $\boxed{\,|\, g(y) - c \,| < \delta_1\,}$.

(This would not give $g(y) \neq c$.)

(Remember: The f-limit needs $\boxed{\,0 < |\, g(y) - c \,| < \delta_1\,}$, in particular, that $g(y) \neq c$.)

Could a change in the properties of f or of g

make these boxed inequalities match?

Yes, if f were continuous, or if g were strictly monotone.

A Limit, $\lim_{x \to c} f(x) = L$, in Terms of δ 's and ϵ 's

For any $\epsilon > 0$, there is a $\delta > 0$ (depending on ϵ, f and c) such that $\boxed{0 < |x - c| < \delta}$ would imply $|f(x) - L| < \epsilon$.

↑ This requires that $x \neq c$.

Continuity, $\lim_{x \to c} f(x) = f(c)$, in Terms of δ 's and ϵ 's

For any $\epsilon > 0$, there is a $\delta > 0$ (depending on ϵ, f and c) such that $\boxed{|x - c| < \delta}$ would imply $|f(x) - f(c)| < \epsilon$.

↑ This doesn't.

This definition does not require $0 < |x - c|$.

Even if we had $0 = |x - c|$, we would then have $x = c$,

which would imply that $f(x) = f(c)$

and $|f(x) - f(c)| = 0 < \epsilon$, anyway.

Suppose now that f were continuous at $x = c$:

$\lim_{x \to c} f(x) = f(c)$ **would now mean that:**

For any $\epsilon_1 > 0$, there would be a $\delta_1 > 0$

such that $|\, x - c \,| < \delta_1$ would imply $|\, f(x) - f(c) \,| < \epsilon_1$.

If x were replaced by $g(y)$, this would become:

For any $\epsilon_1 > 0$, there would be a $\delta_1 > 0$ (depending on ϵ_1, f and c)

such that $\boxed{|\, g(y) - c \,| < \delta_1}$ implies: $|\, f(g(y)) - f(c) \,| < \epsilon_1$,

while $\lim_{y \to d} g(y) = c$ would still mean that:

For our $\delta_1 > 0$ there is a $\delta_2 > 0$

such that $0 < |\, y - d \,| < \delta_2$ implies $\boxed{|\, g(y) - c \,| < \delta_1}$.

The connecting, boxed inequalities would now match.

If f is continuous at $x = c$

and if $\lim_{y \to d\pm} g(y)$ equals this c,

then $\lim_{y \to d\pm} f\big(g(y)\big)$ equals $f\Big(\lim_{y \to d\pm} g(y)\Big)$ equals $f(c)$.

Proof: Combine

$$f(c + o(1)) = f(c) + o(1), \text{ meaning } \lim_{x \to c} f(x) = f(c),$$

with $\quad g(y) = c + o(1), \text{ meaning } \lim_{y \to d} g(y) = c,$

to obtain $f\big(g(y)\big) = f\big(c + o(1)\big) = f(c) + o(1)$. $\square$

Notes: This works whether or not g is continuous at y = d.

The $\pm$'s above mean that the limits in y can be

all left-handed, all right-handed, or all two-sided.

If f is continuous at $x = c$

and if $\lim_{y \to d\pm} g(y)$ equals this c,

then $\lim_{y \to d\pm} f\big(g(y)\big)$ equals $f\left(\lim_{y \to d\pm} g(y)\right)$ equals $f(c)$.

Examples:

$$\text{for } f(x) = \sqrt{x}, \qquad \lim_{y \to d} \sqrt{g(y)} = \sqrt{\lim_{y \to d} g(y)} = \sqrt{c};$$

$$\text{for } f(x) = \ln x, \qquad \lim_{y \to d+} \ln g(y) = \ln \lim_{y \to d+} g(y) = \ln c;$$

$$\text{for } f(x) = e^x, \qquad \lim_{y \to d-} e^{g(y)} = e^{\lim_{y \to d-} g(y)} = e^c;$$

$$\text{for } f(x) = x^2, \qquad \lim_{y \to d\pm} \left(g(y)^2\right) = \left(\lim_{y \to d\pm} g(y)\right)^2 = c^2.$$

If f is continuous at g(d): $\lim_{x \to g(d)} f(x) = f(g(d)),$

and if g is continuous at d: $\lim_{y \to d} g(y) = g(d),$

then f(g(y)) is also continuous at $y = d$.

and we have $\lim_{y \to d} f(g(y)) = f(g(d)).$

Proof: Combine

$$y = d + o(1), \qquad \text{(since } y \longrightarrow d\text{)}$$

$$g(y) = g\big(d + o(1)\big) = g(d) + o(1), \qquad (g \text{ is continuous})$$

and $f\big(g(d) + o(1)\big) = f\big(g(d)\big) + o(1),$

to obtain (since f is continuous)

$$f\big(g(y)\big) = f\Big(g\big(d + o(1)\big)\Big) = f\big(g(d) + o(1)\big) = f\big(g(d)\big) + o(1).$$

If f is continuous at g(d): $$\lim_{x \to g(d)} f(x) = f(g(d)),$$

and if g is continuous at d: $$\lim_{y \to d} g(y) = g(d),$$

then $f(g(y))$ is also continuous at $y = d$.

and we have $$\lim_{y \to d} f(g(y)) = f(g(d)).$$

Examples:

for $f(x) = \sqrt{x}$, $\lim_{y \to d} \sqrt{g(y)} = \sqrt{\lim_{y \to d} g(y)} = \sqrt{g(d)}$;

for $f(x) = x^2$, $\lim_{y \to d} \left(g(y)^2\right) = \left(\lim_{y \to d} g(y)\right)^2 = g(d)^2$.

for $f(x) = \frac{1}{x}$, $\lim_{y \to d} \frac{1}{g(y)} = \frac{1}{\lim_{y \to d} g(y)} = \frac{1}{g(d)}$,

if $g(d) \neq 0$.

Now, let's return to assuming only that f has a right-hand limit at $x = c$:

$$f(c+) = \lim_{x \to c+} f(x) = L \text{ would require that}$$

For any $\epsilon_1 > 0$, there is a $\delta_1 > 0$ (depending on ϵ_1, f and c) such that $0 < \; x - c \; < \delta_1$ would imply $| \, f(x) - L \, | < \epsilon_1$.

If x were replaced by $g(y)$, this would become

For any $\epsilon_1 > 0$, there is a $\delta_1 > 0$ (depending on ϵ_1, f and c) such that $\boxed{0 < g(y) - c < \delta_1}$

would imply $| \, f(g(y)) - L \, | < \epsilon_1$,

(In particular, this would need $g(y) > c$.)

On the other hand, $\lim_{y \to d+} g(y) = c$ would yield

For our $\epsilon_2 = \delta_1 > 0$ there is a $\delta_2 > 0$

(depending on δ_1, g and d)

such that $0 < | y - d | < \delta_2$

would imply only that $\boxed{| g(y) - c | < \delta_1}$.

This alone would not be the same as $\boxed{0 < g(y) - c < \delta_1}$.

Now, if we also had g increasing for y near $d+$, with $y > d$,

$g(y)$ would be greater than $g(d+) = c$

and we would have $\boxed{0 < g(y) - c < \delta_1}$,

exactly as the right-limit for f requires:

$$\boxed{0 < g(y) - c < \delta_1}$$

A One-Sided Limit for Composite Functions:

If g(y) is increasing as y approaches $d+$,

if $c = \lim_{y \to d+} g(y)$ exists, and if $\lim_{x \to c+} f(x)$ also exists,

then $\lim_{y \to d+} f(g(y))$ equals this $\lim_{x \to c+} f(x)$.

Example:

$$\text{for } g(y) = \sqrt{y-d}, \quad \lim_{y \to d+} f\left(\sqrt{y-d}\right) = \lim_{x \to 0+} f(x).$$

This theorem works

whenever the one-sided limits $c = g(d+)$ and $f(c+)$ exist,

whether or not f is continuous,

and whether or not $f(x)$ is even defined at $x = c$.

Another One-Sided Limit for Composite Functions:

If g(y) is increasing as y approaches $d\pm$,

if $c = \lim_{y \to d\pm} g(y)$ exists, and if $\lim_{x \to c\pm} f(x)$ also exists,

then $\lim_{y \to d\pm} f(g(y))$ equals this $\lim_{x \to c\pm} f(x)$.

And Still Another One-Sided Limit for Composite Functions:

If g(y) is decreasing as y approaches $d\pm$,

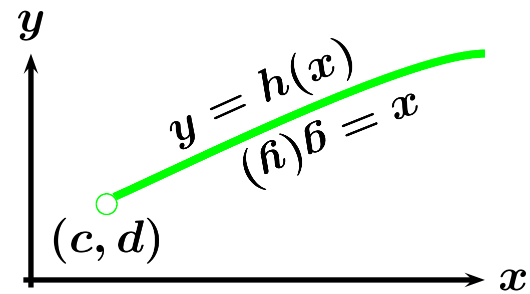
if $c = \lim_{y \to d\pm} g(y)$ exists, and if $\lim_{x \to c\mp} f(x)$ also exists,

then $\lim_{y \to d\pm} f(g(y))$ equals this $\lim_{x \to c\mp} f(x)$.

If $g(y)$ were increasing and, moreover, continuous on an interval on either($\pm$) side of $y = d$, then the function g would have an inverse function h there, such that $h(g(y)) = y$ and $g(h(x)) = x$.

Assume, as before, that $\lim_{y \to d\pm} g(y) = c$.

Then we also have $\lim_{x \to c\pm} h(x) = d$.



We have seen that the existence of $\lim_{x \to c\pm} f(x)$ would imply the existence of $\lim_{y \to d\pm} f(g(y))$, and their equality: $\lim_{y \to d\pm} f(g(y)) = \lim_{x \to c\pm} f(x)$.

Conversely, if $\lim_{y \to d\pm} f(g(y))$ were known to exist, we could have $\lim_{x \to c\pm} f(x) = \lim_{x \to c\pm} f(g(h(x))) = \lim_{y \to d\pm} f(g(y))$ similarly.

Two Equivalences for One-Sided Limits of Composite Functions:

If g(y) is increasing and continuous as y approaches $d\pm$, with $c = \lim_{y \to d\pm} g(y)$, then the limits $\lim_{x \to c\pm} f(x)$ and $\lim_{y \to d\pm} f(g(y))$ either both exist, and are equal, or both limits do not exist.

If g(y) is decreasing and continuous as y approaches $d\pm$, with $c = \lim_{y \to d\pm} g(y)$, then the limits $\lim_{x \to c\mp} f(x)$ and $\lim_{y \to d\pm} f(g(y))$ either both exist, and are equal, or both limits do not exist.

Note that g does not have to be defined or continuous at $y = d$. $c = \lim_{y \to d\pm} g(y) = g(d\pm)$ would not have to equal any $g(d)$.

An Equivalence for Two-Sided Limits of Composite Functions:

If $g(y)$ is monotone and continuous for $d-\delta < y < d+\delta$. Then the limits $\lim\limits_{x \to g(d)} f(x)$ and $\lim\limits_{y \to d} f\big(g(y)\big)$ either both exist, and are equal, or both limits do not exist.

These equivalences justify limit substitutions: $\begin{cases} x = g(y), \\ y = g^{-1}(x), \end{cases}$

$$\lim_{y \to d} f\big(g(y)\big) \;=\; \lim_{x \to \lim\limits_{y \to d} g(y)} f\Big(g\big(g^{-1}(x)\big)\Big) \;=\; \lim_{x \to g(d)} f(x),$$

$$\lim_{y \to d} f\big(g(y)\big) \;=\; \lim_{y \to \lim\limits_{x \to g(d)} g^{-1}(x)} f\big(g(y)\big) \;=\; \lim_{x \to g(d)} f(x).$$

These equivalences justify limit substitutions: $\begin{cases} x = g(y), \\ y = g^{-1}(x), \end{cases}$

$$\lim_{y \to d} f\big(g(y)\big) = \lim_{x \to \lim\limits_{y \to d} g(y)} f\big(g\big(g^{-1}(x)\big)\big) = \lim_{x \to c} f(x).$$

Example: Using $\begin{cases} x = g(y) = y^2, \\ y = g^{-1}(x) = \sqrt{x}, \end{cases}$ we have

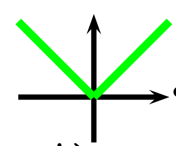
$$\lim_{y \to \sqrt{\pi}} \cos y^2 = \lim_{x \to \lim\limits_{y \to \sqrt{\pi}} y^2} \cos\left(\sqrt{x}\right)^2 = \lim_{x \to \pi} \cos x = -1,$$

$$\lim_{x \to \pi^2} \sin \sqrt{x} = \lim_{y \to \lim\limits_{x \to \pi^2} \sqrt{x}} \sin \sqrt{y^2} = \lim_{y \to \pi} \sin y = 0.$$

Example: Using $\begin{cases} n = m + 1, \\ m = n - 1, \end{cases}$ $\lim\limits_{m \to \infty} a_{m+1} = \lim\limits_{n \to \lim\limits_{m \to \infty} m+1} a_n = \lim\limits_{n \to \infty} a_n$

If $g(y)$ is monotone and continuous for $d - \delta < y < d + \delta$.
Then the limits $\lim_{x \to g(d)} f(x)$ and $\lim_{y \to d} f\big(g(y)\big)$
either both exist, and are equal, or both limits do not exist.

Warning: If the function g were not monotone on both sides of $y = d$, $\lim_{x \to g(d)} f(x)$ could imply $\lim_{y \to d} f\big(g(y)$, but not $\Longleftarrow$.

Consider $g(y) = |y|$ on both sides of $y = 0$,  **.**
Existence of $\lim_{x \to 0} f(x)$ implies existence of $\lim_{y \to 0} f(|y|)$,
but existence of $\lim_{y \to 0} f(|y|)$ does not involve f(negative x).
However, $\lim_{x \to 0+} f(x) = \lim_{y \to 0+} f(|y|)$,
since $|y|$ is increasing for $y > 0$.

These equivalences justify limit substitutions: $\begin{cases} h = g(u), \\ u = g^{-1}(h), \end{cases}$

$$\lim_{u \to d} F\big(g(u)\big) = \lim_{h \to \lim\limits_{u \to d} g(u)} F\Big(g\big(g^{-1}(h)\big)\Big) = \lim_{h \to c} F(h).$$

Example: With x a constant and using $\begin{cases} h = g(u) = u - x, \\ u = g^{-1}(h) = x + h, \end{cases}$

we have $\displaystyle \lim_{u \to x} \frac{f(u) - f(x)}{u - x} = \lim_{h \to \lim\limits_{u \to x}(u - x)} \frac{f(x+h) - f(x)}{(x+h) - x}$

$$\Uparrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\lim_{u \to \lim\limits_{h \to 0}(x+h)} \frac{f(x + (u - x)) - f(x)}{u - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substitution as an Alternative to Factoring

If we let $\begin{cases} x = y + 3 \\ y = x - 3 \end{cases}$, we have

$$\lim_{x\to 3} \frac{x^2-9}{x^2-x-6} = \lim_{y\to \lim\limits_{x\to 3}(x-3)} \frac{(y+3)^2-9}{(y+3)^2-(y+3)-6}$$

$$= \lim_{y\to 0} \frac{y^2+6y+9-9}{y^2+6y+9-y-3-6}$$

$$= \lim_{y\to 0} \frac{y^2+6y}{y^2+5y}$$

$$= \lim_{y\to 0} \frac{y+6}{y+5} = \frac{6}{5}.$$

$$\lim_{x \to 1} \frac{x^p - 1}{x - 1} = p, \text{ for any integer } p$$

Proof:

If $p > 0$, we start with

$$\begin{aligned}(x-1)(x^{p-1} + x^{p-2} + \ldots + x + 1) &= x^p + x^{p-1} + \ldots + x^2 + x \\ &\quad - x^{p-1} - x^{p-2} - \ldots - x - 1 \\ &= x^p - 1.\end{aligned}$$

The desired limit now becomes

$$\begin{aligned}\lim_{x \to 1} \frac{x^p - 1}{x - 1} &= \lim_{x \to 1} x^{p-1} + x^{p-2} + \ldots + x + 1 && (p \text{ terms}) \\ &= 1 + 1 + \ldots + 1 + 1 && (p \text{ ones}) \\ &= p.\end{aligned}$$

For $p = 0$, the limit is much simpler:

$$\lim_{x\to 1} \frac{x^p - 1}{x - 1} = \lim_{x\to 1} \frac{x^0 - 1}{x - 1} = \lim_{x\to 1} \frac{1 - 1}{x - 1} = \lim_{x\to 1} 0 = 0 = p.$$

For $p < 0$, $\quad \displaystyle\lim_{x\to 1} \frac{x^p - 1}{x - 1} = \lim_{u\to 1} \frac{\left(\frac{1}{u}\right)^p - 1}{\frac{1}{u} - 1}$, using $\begin{cases} x = \frac{1}{u}, \\ u = \frac{1}{x}, \end{cases}$

$$= \lim_{u\to 1} u \frac{u^{-p} - 1}{1 - u} = \lim_{u\to 1} (-u) \frac{u^{-p} - 1}{u - 1} = (-1)(-p) = p,$$

by an application of the first part, since $-p$ is positive. $\square$

$$\boxed{\lim_{y\to 1}\frac{y^{\frac{p}{q}}-1}{y-1} \;=\; \frac{p}{q},\ \text{for any rational } \frac{p}{q}.}$$

Proof: If we use the previous result, we have

$$\begin{aligned}
\lim_{y\to 1}\frac{y^{\frac{p}{q}}-1}{y-1} &= \lim_{y\to 1}\frac{y^{\frac{p}{q}}-1}{y^{\frac{q}{q}}-1} \\
&= \lim_{x\to 1}\frac{x^{p}-1}{x^{q}-1}, \qquad \text{substituting } \begin{cases} y = x^{q}, \\ x = y^{\frac{1}{q}}, \end{cases} \\
&= \frac{\lim_{x\to 1}\frac{x^{p}-1}{x-1}}{\lim_{x\to 1}\frac{x^{q}-1}{x-1}} \;=\; \frac{p}{q}. \quad \square
\end{aligned}$$

$$\frac{d}{dx}\; x^{\frac{p}{q}} \;=\; \frac{p}{q}\; x^{\frac{p}{q}-1},\ \text{unless } \left(x = 0 \text{ and } \frac{p}{q} \leq 1\right).$$

Proof: For nonzero x, the difference quotient has the limit

$$\lim_{h\to 0} \frac{(x+h)^{\frac{p}{q}} - x^{\frac{p}{q}}}{h} = \lim_{u\to x} \frac{u^{\frac{p}{q}} - x^{\frac{p}{q}}}{u - x} = \lim_{u\to x} \;\frac{x^{\frac{p}{q}}\left(\dfrac{u^{\frac{p}{q}}}{x^{\frac{p}{q}}} - 1\right)}{x\left(\dfrac{u}{x} - 1\right)},$$

so, if we substitute v for $\frac{u}{x}$, we have

$$= \frac{x^{\frac{p}{q}}}{x} \cdot \lim_{v\to \lim\limits_{u\to x} \frac{u}{x}} \frac{v^{\frac{p}{q}} - 1}{v - 1} = x^{\frac{p}{q}-1} \lim_{v\to 1} \frac{v^{\frac{p}{q}} - 1}{v - 1} = x^{\frac{p}{q}-1}\; \frac{p}{q}.$$

When x equals zero, and when $\frac{p}{q}$ is greater than 1, we can verify

that $\lim\limits_{h\to 0} \dfrac{(0+h)^{\frac{p}{q}} - 0^{\frac{p}{q}}}{h} \;=\; \lim\limits_{h\to 0} h^{\frac{p}{q}-1} \;=\; 0 \;=\; \dfrac{p}{q}\cdot 0^{\frac{p}{q}-1}.\quad \square$