There are many important limits

which cannot be evaluated directly,

but which can be guaranteed to exist,

as Least Upper Bounds,

or as Greatest Lower Bounds.

Here is a first example:

Every function $F(x)$ which is nondecreasing as $x \longrightarrow b-$
either diverges to infinity: $\lim_{x \to b-} F(x) = \infty.$
or converges to some finite limit L : $\lim_{x \to b-} F(x) = L.$

Some Definitions:

$$F \text{ is said to be increasing if } F(x) < F(y) \text{ whenever } x < y.$$

$$F \text{ is said to be nondecreasing if } F(x) \leq F(y) \text{ whenever } x < y.$$

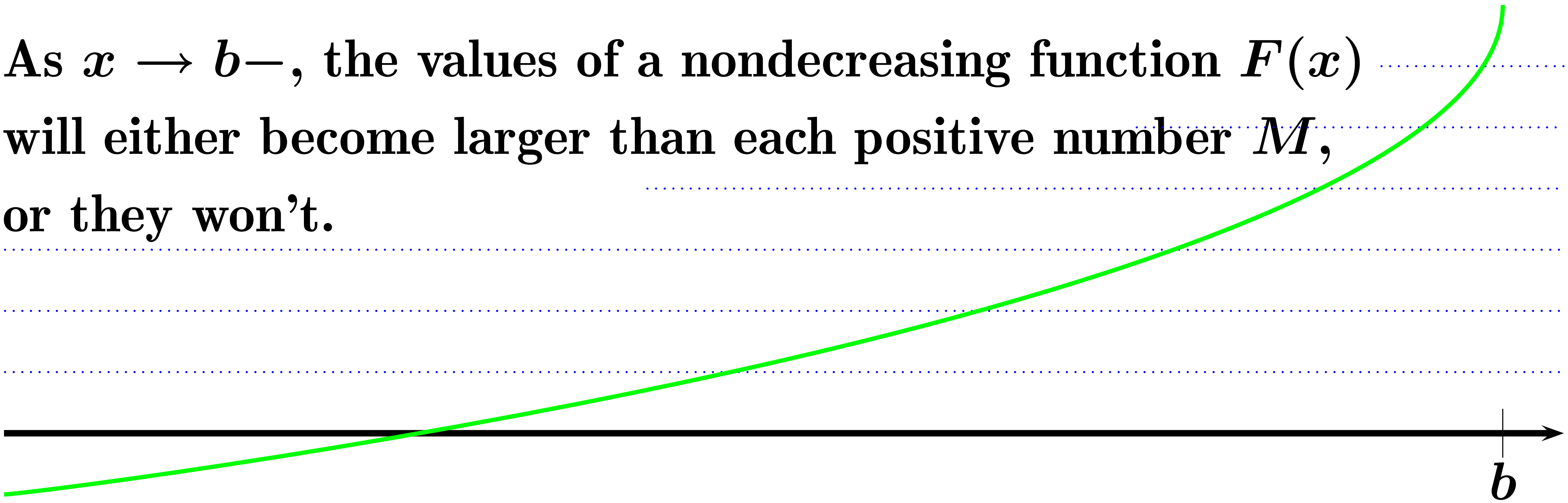
$$F \text{ is said to be nonincreasing if } F(x) \geq F(y) \text{ whenever } x < y.$$

$$F \text{ is said to be decreasing if } F(x) > F(y) \text{ whenever } x < y.$$

Every increasing function is also nondecreasing.

Every decreasing function is also nonincreasing.

As $x \longrightarrow b-$, the values of a nondecreasing function $F(x)$ will either become larger than each positive number M, or they won't.



If $F(x)$ were to become larger than each finite number M, then for each M

there would be at least one x_M such that $F(x_M) > M$.

Moreover for all x to the right of x_M we would have

$f(x) \geq f(x_M) > M$, because F is nondecreasing.

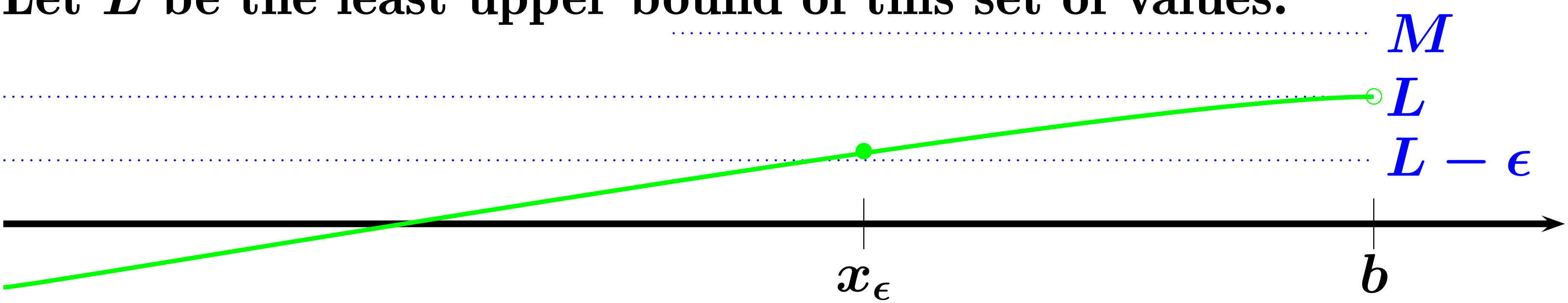
This is precisely what it would mean to have $\boxed{\lim_{x \to b-} F(x) = \infty.}$

On the other hand, if there were a number M

which was always greater than $F(x)$,

then M would be an upper bound on the values of $y = F(x)$.

Let L be the least upper bound of this set of values.



For any positive ϵ,

the number $L - \epsilon$ would be smaller than L

so that $L - \epsilon$ would not be an upper bound.

It would be surpassed somewhere, say at x_ϵ : $F(x_\epsilon) > L - \epsilon$.

For any $x > x_\epsilon$ we would have $L - \epsilon < F(x_\epsilon) \leq F(x) \leq L$.

This would mean that $\boxed{\lim_{x \to b-} F(x) = L.}$

The theorem above also works as $x \longrightarrow \infty$:

Every function $F(x)$
which is nondecreasing as $x \longrightarrow \infty$
either diverges to infinity: $\lim_{x\to\infty} F(x) = \infty.$
or converges to some finite limit L : $\lim_{x\to\infty} F(x) = L.$

By reversing some of the inequalities above

we can also prove that:

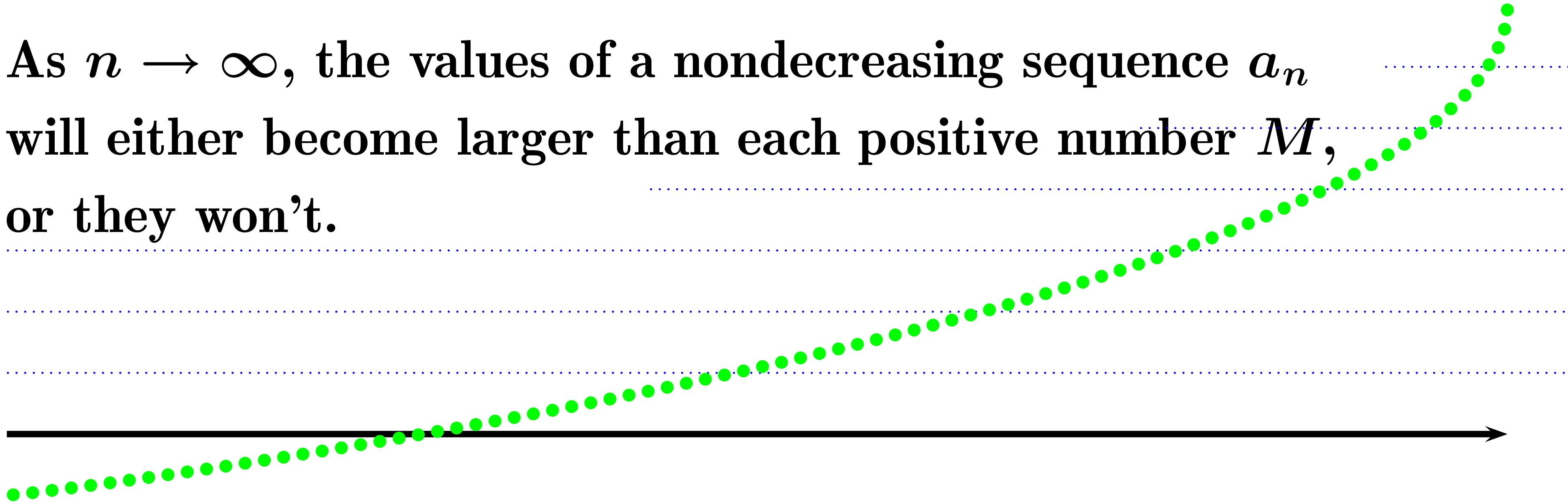
Every function $F(x)$
which is nonincreasing as $x \longrightarrow \infty$
either diverges to $-\infty$: $\lim_{x\to\infty} F(x) = -\infty.$
or converges to some finite limit L : $\lim_{x\to\infty} F(x) = L.$

These theorems also work for sequences:

Every nondecreasing sequence
$a_0 \le a_1 \le a_2 \le a_3 \le a_4 \le \ldots \le a_n \le \ldots$
either diverges to infinity: $\lim_{n\to\infty} a_n = \infty.$
or converges to some finite limit L **:** $\lim_{n\to\infty} a_n = L.$

Every nonincreasing sequence
$a_0 \ge a_1 \ge a_2 \ge a_3 \ge a_4 \ge \ldots \ge a_n \ge \ldots$
either diverges to $-\infty$**:** $\lim_{x\to\infty} a_n = -\infty.$
or converges to some finite limit L **:** $\lim_{n\to\infty} a_n = L.$

As $n \to \infty$, the values of a nondecreasing sequence a_n will either become larger than each positive number M, or they won't.



If a_n were to become larger than each finite number M, then for each M

there would be at least one n_M such that $a_{n_M} > M$.

Moreover for all n beyond n_M we would have

$a_n \geq a_{n_M} > M$, because a_n is nondecreasing.

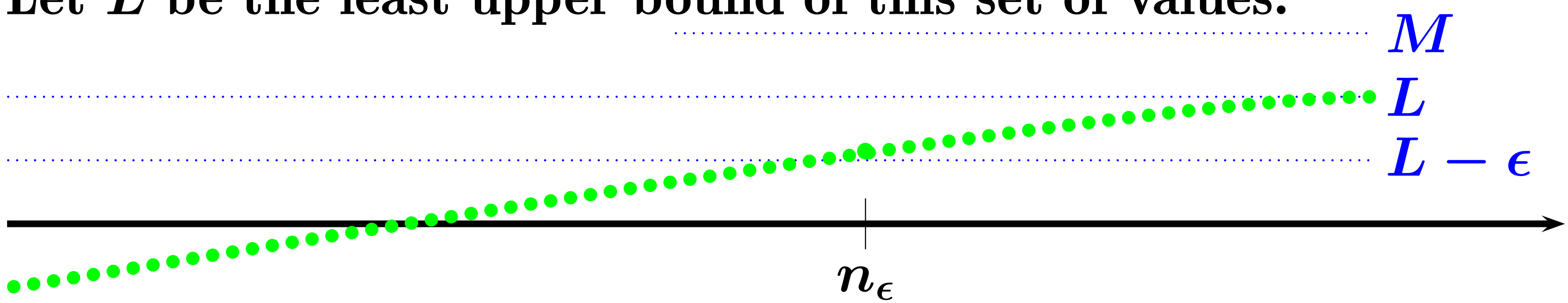
This is precisely what it would mean to have $\boxed{\lim_{n\to\infty} a_n = \infty.}$

On the other hand, if there were a number M

which was always greater than a_n,

then M would be an upper bound on the values of a_n.

Let L be the least upper bound of this set of values.



For any positive ϵ,

the number $L - \epsilon$ would be smaller than L

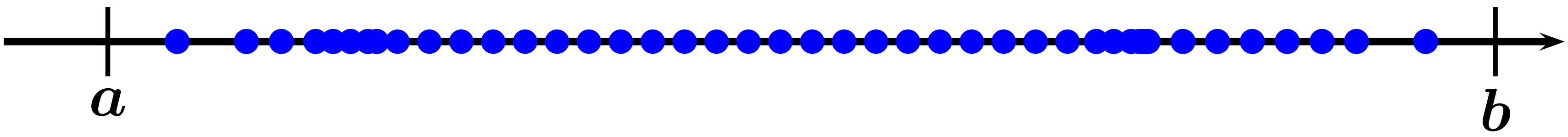
so that $L - \epsilon$ would not be an upper bound.

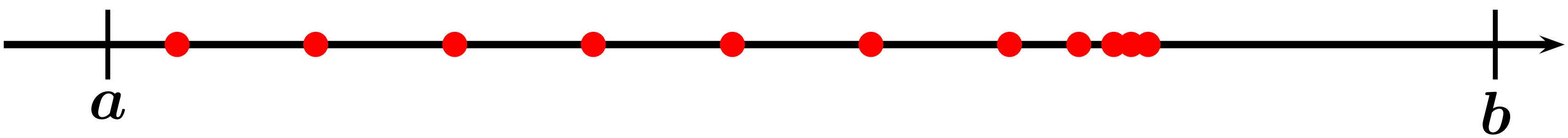
It would be surpassed somewhere, say at n_ϵ : $\quad a_{n_\epsilon} > L - \epsilon$.

For any $n > n_\epsilon$ we would have $L - \epsilon < a_{n_\epsilon} \leq a_n \leq L$.

This would mean that $\boxed{\lim_{n\to\infty} a_n = L.}$

Any infinite set contained in a finite closed interval contains a distinct subsequence converging to a limit within that interval.





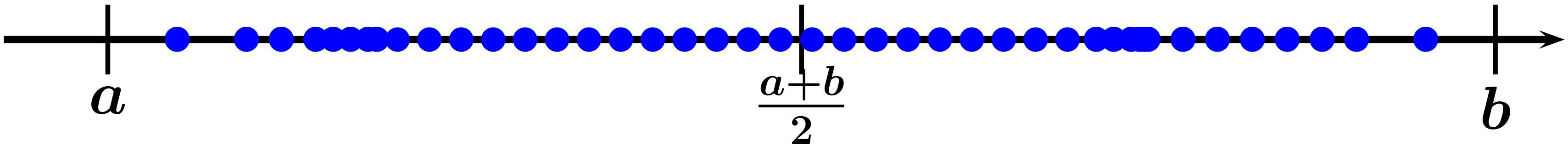
Remark:

This will be useful in situations where we need to know that a convergent sequence of certain numbers exists, even if we won't know where its limit is.

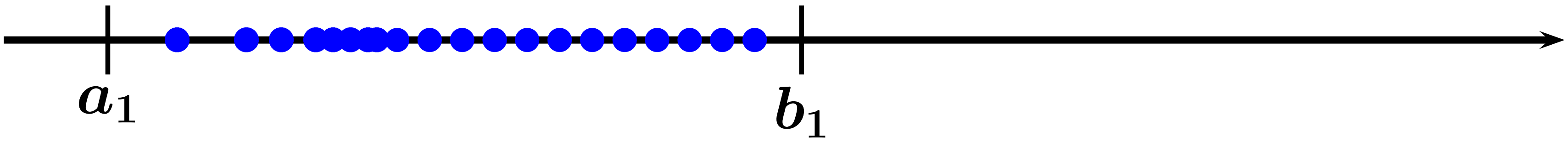
Any infinite set contained in a finite closed interval contains a distinct subsequence converging to a limit within that interval.

Proof: We shall bisect this interval over and over.

Consider this subdivision:

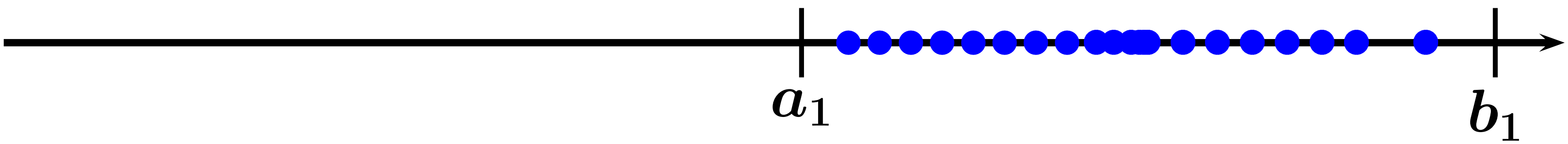


If an infinite subset of the original subset lies in the left half, let $a_1 = a$ and let $b_1 = \frac{a+b}{2}$:

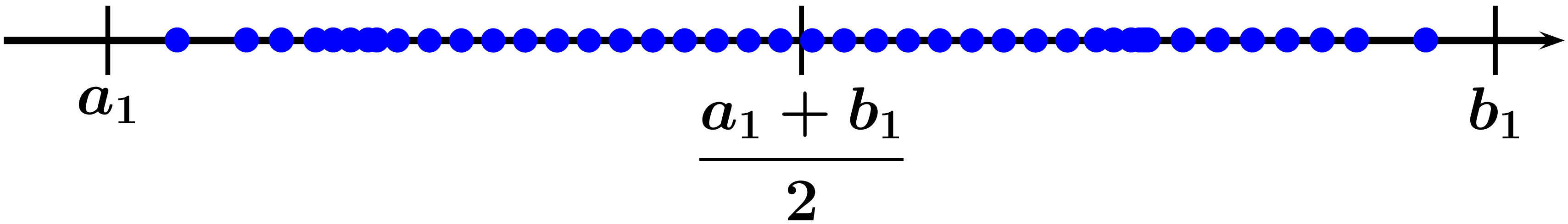


Otherwise the right half contains an infinite subset.

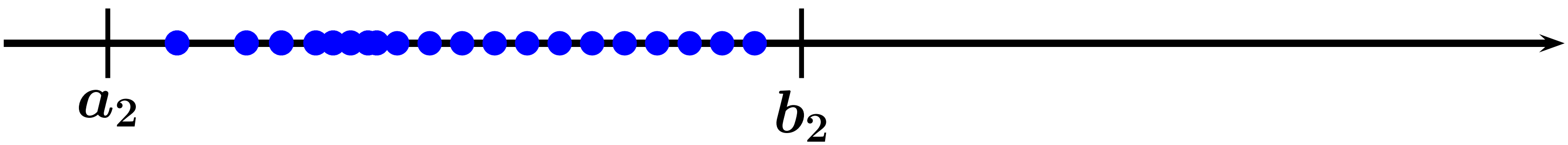
If so, let $a_1 = \dfrac{a+b}{2}$ and let $b_1 = b$:



Now consider this subdivision:

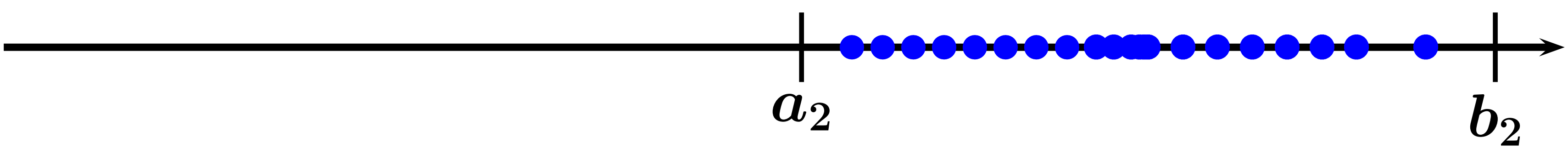


If an infinite subset of the original subset lies in the left half, let $a_2 = a_1$ and let $b_2 = \frac{a_1 + b_1}{2}$:

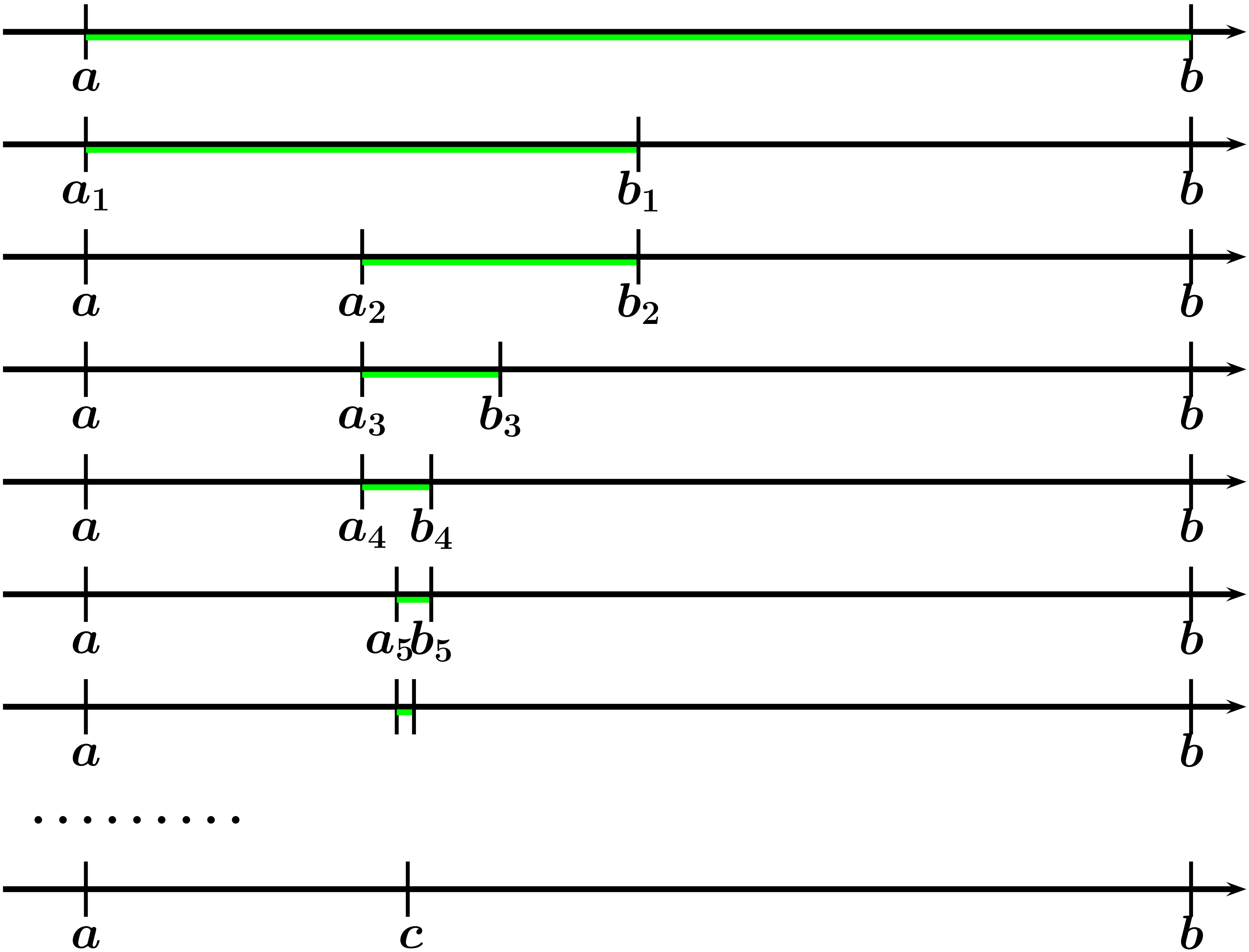


Otherwise the right half contains an infinite subset.

If so, let $a_2 = \frac{a_1 + b_1}{2}$ and let $b_2 = b_1$:



and so on......



a
b
a1
b1
b
a
a2
b2
b
a
a3
b3
b
a
a4
b4
b
a
a5
b5
b
a
b
. . . .
a
c
b

We have an infinite sequence of intervals $[a_n, b_n]$,

each a right or left half of the interval before,

and each containing an infinite subset of the original set.

$$a \le a_1 \le a_2 \le \ldots \le a_n \le \ldots \le b_n \le \ldots \le b_2 \le b_1 \le b$$

The a_n's are nondecreasing and are bounded above by b,

so they converge to a limit, their least upper bound.

The b_n's are nonincreasing and are bounded below by a,

so they converge to a limit, their greatest lower bound.

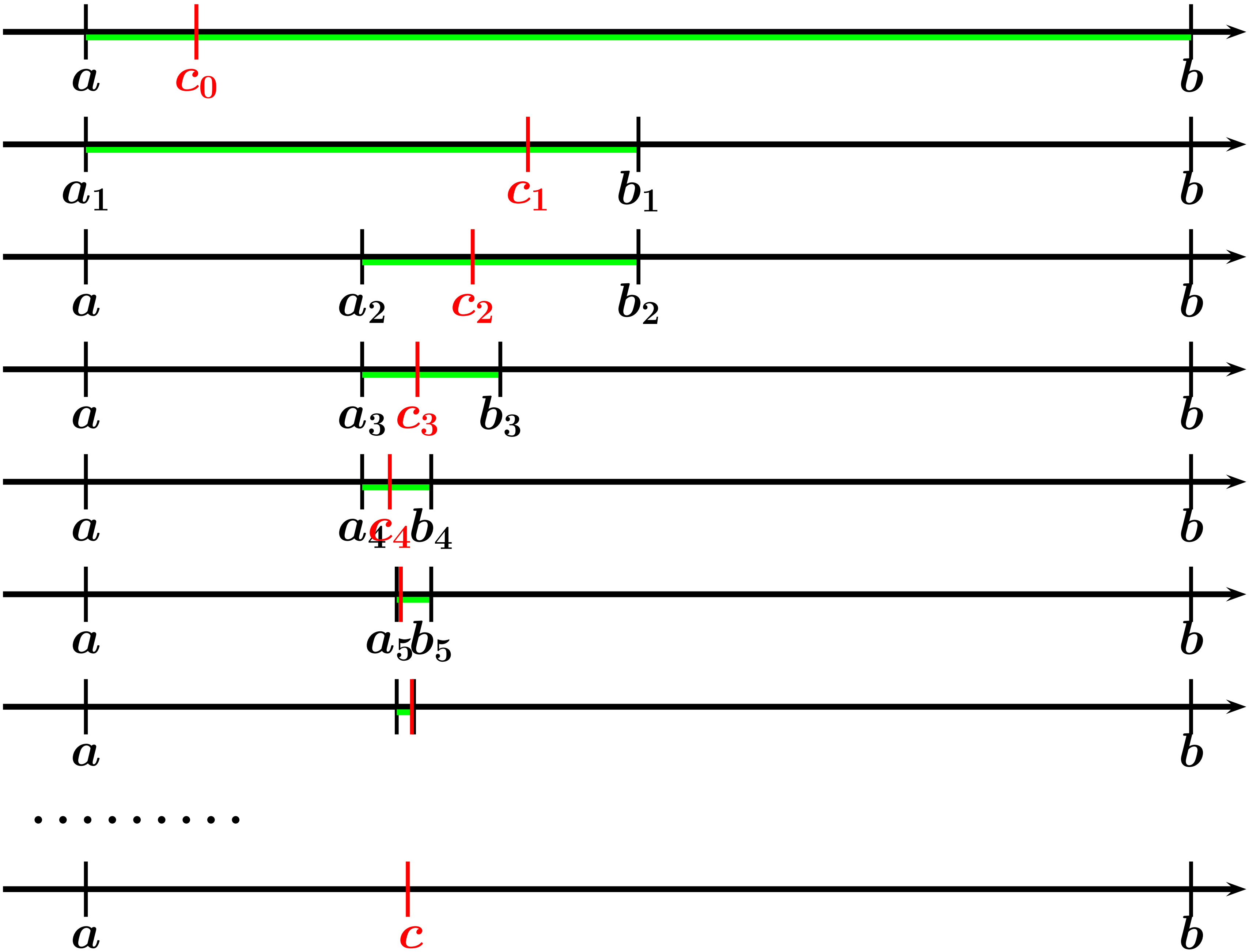
Since $\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^n} = 0$,

we have $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = c$, **where** $a \le c \le b$.

So far, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$, where $a \leq c \leq b$.

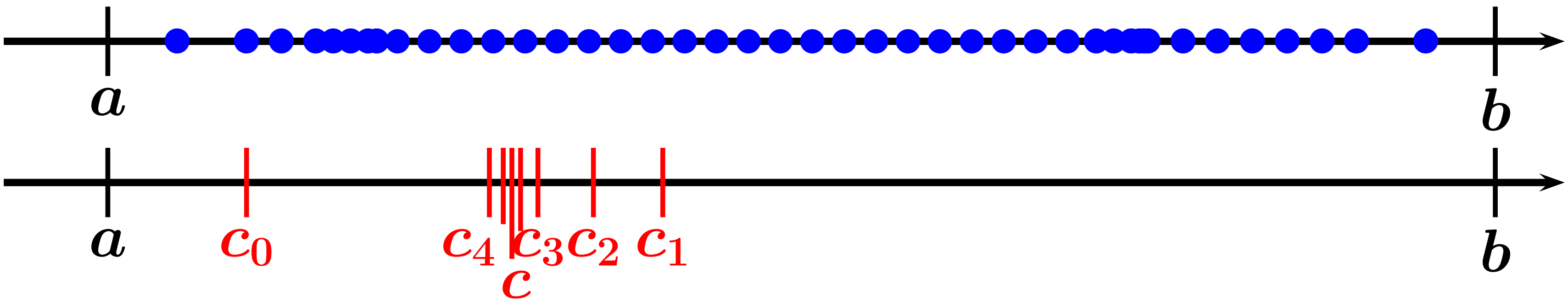
Now, from the original infinite subset,
we shall select a sequence of distinct c_n's.
For each $n = 1, 2, \ldots,$ each subinterval $[a_n, b_n]$
contains an infinite subset of the original set.
We select a c_n from this infinite subset,
but we make sure that it is a not the same as $c_1, c_2, \ldots$ or c_{n-1}.

Since each c_n satisfies $a_n \leq c_n \leq b_n$,
and since we already have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$,
the c_n's also converge to this c, where $a \leq c \leq b$.



a c_0 b
a_1 c_1 b_1 b
a a_2 c_2 b_2 b
a a_3 c_3 b_3 b
a a_4 c_4 b_4 b
a a_5 b_5 b
a b
a c b

Bolzano(1806)-Weierstrass(1855) Theorem:
Any infinite set contained in a finite closed interval contains a distinct subsequence converging to a limit within that interval.



Another Remark:

This will make the following facts easy to prove about the behavior of any function that is continuous on a closed interval of finite length on the real line.

If $f(x)$ is a continuous function for real $x : a \leq x \leq b$, then $f(x) \leq M$ for some finite number M.

Significance: This function doesn't shoot up infinitely high.

Proof by Contradiction:

If this theorem weren't true, then, for each positive integer n,

there would be some x_n

where $f(x_n) \geq n$ and $f(x_n) > f(x_{n-1})$.

We would have $\lim_{n \to \infty} f(x_n) = \infty$.

The Bolzano-Weierstrass Theorem picks out of these x_n's a subsequence $\{x_{n_k}\}$ approaching some c in $[a, b]$.

The continuity of $f(x)$ at c would imply $\lim_{k \to \infty} f(x_{n_k}) = f(c)$.

If $f(x)$ is a continuous function for real $x : a \leq x \leq b$,
then $f(x) \leq f(c)$ for some real number c in $[a, b]$.

Significance: This function has a maximum value.

Proof:

The last result gave an upper bound M

for the set of values of $y = f(x)$ in $a \leq x \leq b$.

There must also be a least upper bound L for these values.

For each positive integer n, there is an x_n

satisfying $L - \dfrac{1}{n} \leq f(x_n) \leq L$, and $a \leq x_n \leq b$.

If any $f(x_n)$ were to equal this L, the desired c could be x_n,

and we would have any $f(x) \leq L = f(c)$.

Otherwise, there would be an infinite set of different x_n's, and the Bolzano-Weierstrass Theorem could pick out of these x_n's a subsequence $\{x_{n_k}\}$ approaching some c in $[a, b]$.

The continuity of $f(x)$ at this c would give

$$\begin{aligned} L - \frac{1}{n_k} \leq & \quad f(x_{n_k}) \leq L, \\ \lim_{k\to\infty}\left(L - \frac{1}{n_k}\right) \leq & \lim_{k\to\infty} f(x_{n_k}) \leq L, \\ L - 0 \leq & f(\lim_{k\to\infty} x_{n_k}) \leq L, \\ L \leq & f(c) \qquad \leq L, \end{aligned}$$

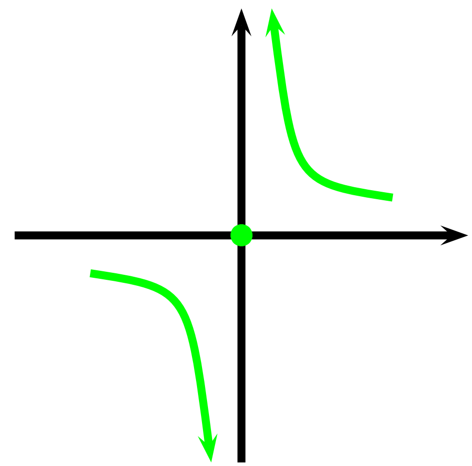
so that $f(c)$ would equal L,

and we would have any $f(x) \leq L = f(c)$.

Another Remark:

Just to show how important it is for a function to be [continuous] [on a closed interval] [of finite length] [on the real line], in these last results, consider these borderline examples:

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{for } -1 \le x < 0, \\ 0 & \text{for } \quad x = 0, \\ \dfrac{1}{x} & \text{for } \quad 0 < x \le 1. \end{cases}$$

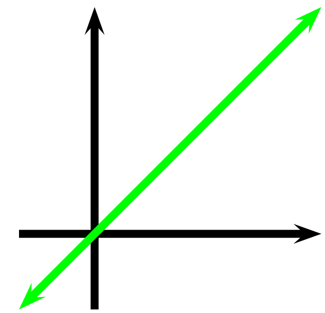


The function f is defined on the [finite] [closed] interval $[-1, 1]$, but it is [not continuous] at $x = 0$.

It has neither an upper bound nor a maximum.

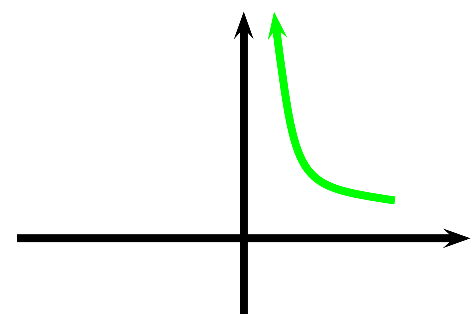
$g(x) = \ x$

is continuous on $-\infty < x < \infty$, which is not a finite interval.



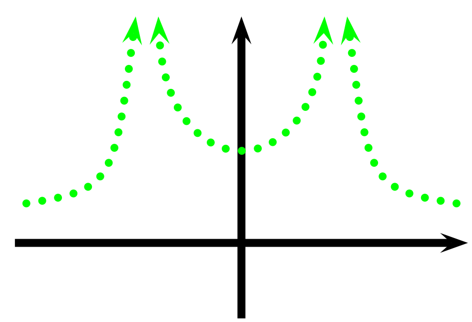
$h(x) = \dfrac{1}{x}$,

which is continuous on $0 < x < 1$, a finite, but not closed, interval.



$r(x) = \dfrac{1}{|x^2 - 3|}$, for rational x

which is continuous on $-2 \leq x \leq 2$, a finite, closed, interval.



None of these functions has an upper bound nor a maximum.

To have $f(x)$ be continuous at $x = x_0$, $\lim_{x \to x_0} f(x) = f(x_0)$.
usually means that:
For any $\epsilon > 0$ there is a $\delta > 0$, which depends on f, ϵ and x_0,
such that $|y - x_0| < \delta$ would imply that $|f(y) - f(x_0)| < \epsilon$.

Sometimes it would be simpler to have a single δ
which which satisfies this condition at all points x,
not just at $x = x_0$,
i.e. a δ which depends only on f and ϵ.

We shall say that a function $f(x)$ is uniformly continuous iff
for any $\epsilon > 0$ there is a $\delta > 0$, which depends only on f and ϵ
such that $|y - x| < \delta$ for any x and y
would imply that $|f(y) - f(x)| < \epsilon$.

If $f(x)$ is continuous for real $x : a \le x \le b$,
then $f(x)$ is uniformly continuous for $a \le x \le b$.

Proof by Contradiction:

If uniform continuity were not true,

there would be an $\epsilon > 0$ and x_n's and y_n's in $[a, b]$ satisfying

$$|y_n - x_n| < \delta = \frac{1}{n}, \text{ but } |f(y_n) - f(x_n)| \ge \epsilon \text{ for each } n > 0.$$

The Bolzano-Weierstrass Theorem would pick out of these x_n
a subsequence $\{x_{n_k}\}$ approaching a number c in $[a, b]$.

Because $|y_{n_k} - x_{n_k}| < \dfrac{1}{n_k}$, the y_{n_k} would also approach c.

Continuity of f and the assumption that $|f(y_{n_k}) - f(x_{n_k})| \ge \epsilon$
would give us $|f(c) - f(c)| \ge \epsilon$, i.e. that $0 \ge \epsilon$.

In summary, a function which is continuous on a closed interval of finite length on the real line. has some very surprising and powerful properties:

It has an upper bound.

It has at least one maximum point.

It is uniformly continuous.

$$\textbf{For } -1 < x < 1, \lim_{n\to\infty} x^n = 0$$

Examples:

For $x = \frac{1}{2}$, $\{x^n\} = \left\{\left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2^n}\right\}$

$$= \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024}, \cdots\right\}$$

For $x = -\frac{1}{2}$, $\{x^n\} = \left\{\left(-\frac{1}{2}\right)^n\right\} = \left\{\frac{(-1)^n}{2^n}\right\}$

$$= \left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \frac{1}{64}, -\frac{1}{128}, \frac{1}{256}, \cdots\right\}$$

Proof:

First, to show that $\lim_{n\to\infty} x^n = 0$ **for** $0 < x < 1$,

note that $1 > x > x^2 > x^3 > x^4$, **etc.**

The sequence $1, x, x^2, \ldots, x^n, \ldots$ **is decreasing and** > 0.

$\lim_{n\to\infty} x^n$ **converges.**

$$x \lim_{n\to\infty} x^n = \lim_{n\to\infty} x \lim_{n\to\infty} x^n = \lim_{n\to\infty} x^{n+1} = \lim_{n\to\infty} x^n.$$

$$x \lim_{n\to\infty} x^n - \lim_{n\to\infty} x^n = 0.$$

$$(x-1) \lim_{n\to\infty} x^n = 0.$$

$$\lim_{n\to\infty} x^n = 0.$$

To show that $\lim_{n\to\infty} x^n = 0$ **for** $-1 < x < 0$ **:**

For even $n = 2k$,

$$\lim_{\text{even } n\to\infty} x^n = \lim_{k\to\infty} x^{2k} = \lim_{k\to\infty} (x^2)^k = 0,$$

since $0 < x^2 < 1$.

For odd $n = 2k + 1$,

$$\lim_{\text{odd } n\to\infty} x^n = \lim_{k\to\infty} x^{2k+1} = \lim_{k\to\infty} (x^2)^k \cdot x = 0 \cdot x = 0,$$

since $0 < x^2 < 1$.

Finally for $x = 0$ **:**

$$\lim_{n\to\infty} x^n = \lim_{n\to\infty} 0^n = \lim_{n\to\infty} 0 = 0.$$

$$\boxed{\text{For } -1 < x < 1,\ \lim_{n\to\infty} x^n = 0}$$

$$\boxed{\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \text{ for } -1 < x < 1,}$$

Examples:

$$\sum_{k=0}^{\infty} 0^k = 1 + 0 + 0 + 0 + 0 + 0 + \cdots = \frac{1}{1-0} = 1.$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = \frac{1}{1-\frac{1}{2}} = 2.$$

$$\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}.$$

Proof of $\boxed{\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \text{ for } -1 < x < 1.}$

We shall write

$$\begin{aligned}(1-x)(1+x+\ldots+x^{n-2}+x^{n-1}) &= 1+x+\ldots+x^{n-2}+x^{n-1} \\ &\quad -x-x^2-\ldots-x^{n-1}-x^n \\ &= 1 \qquad -x^n\end{aligned}$$

as: $$\sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}.$$

Then, we take the limit as n **approaches** ∞ **:**

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots = \frac{1-0}{1-x}.$$

$$\boxed{\textbf{For } 0 < x,\ \lim_{n\to\infty} x^{\frac{1}{n}} = 1}$$

Examples:

$$2 = 2.000000000000000 \qquad \frac{1}{2} = 0.500000000000000$$

$$\sqrt{2} = 1.414213562373095 \qquad \sqrt{\frac{1}{2}} = 0.707106781186547$$

$$\sqrt[3]{2} = 1.259921049894873 \qquad \sqrt[3]{\frac{1}{2}} = 0.793700525984099$$

$$\sqrt[4]{2} = 1.189207115002721 \qquad \sqrt[4]{\frac{1}{2}} = 0.840896415253714$$

$$\sqrt[5]{2} = 1.148698354997035 \qquad \sqrt[5]{\frac{1}{2}} = 0.870550563296124$$

$$\sqrt[100]{2} = 1.006955550056719 \qquad \sqrt[100]{\frac{1}{2}} = 0.993092495437036$$

$$\text{For } 0 < x, \lim_{n\to\infty} x^{\frac{1}{n}} = 1$$

Proof:

For each positive integer n,

we know that the n-th root function $x^{\frac{1}{n}}$ exists

as the inverse function of the n-th power function x^n.

Since the n-th power function x^n is continuous and increasing

for $x > 0$,

so is the n-th root function $x^{\frac{1}{n}}$.

Assume, WOLG*, that $x > 1$. Then we have $x^{\frac{1}{n}} > 1^{\frac{1}{n}} = 1$.

$$\text{Since } \frac{x^{\frac{1}{n}}}{x^{\frac{1}{n+1}}} = x^{\frac{1}{n}-\frac{1}{n+1}} = x^{\frac{1}{n(n+1)}} > 1, \text{ we have } x^{\frac{1}{n}} > x^{\frac{1}{n+1}}.$$

***Without Loss Of Generality (This cuts our work here by 40%.)**

For $x > 1$, we have a decreasing sequence
$x > x^{\frac{1}{2}} > x^{\frac{1}{3}} > x^{\frac{1}{4}} > \cdots > x^{\frac{1}{n}} > \cdots$ **bounded below by 1.**

$\lim_{n\to\infty} x^{\frac{1}{n}}$, **therefore, exists; and** $\lim_{n\to\infty} x^{\frac{1}{n}} \geq 1$.

This limit is equal to its own square:

$$\left(\lim_{n\to\infty} x^{\frac{1}{n}}\right)^2 = \lim_{n\to\infty} \left(x^{\frac{1}{n}}\right)^2 = \lim_{\text{even } n\to\infty} x^{\frac{2}{n}} = \lim_{2m\to\infty} x^{\frac{2}{2m}} =$$

$= \lim_{m\to\infty} x^{\frac{1}{m}} = \lim_{n\to\infty} x^{\frac{1}{n}}$, **and must equal 1 or 0.**

For $x > 1$, we now have $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$.

For $0 < x < 1$, $\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{x}\right)^{\frac{1}{n}}} = \frac{1}{1} = 1$.

For $x = 1$, $\lim_{n\to\infty} 1^{\frac{1}{n}} = \lim_{n\to\infty} 1 = 1$.

For $0 < x$, $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$

Another Proof of $\boxed{\textbf{For } 0 < x,\ \lim_{n\to\infty} x^{\frac{1}{n}} = 1}$

$$\begin{aligned}(x^{\frac{1}{n}} - 1)(x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}} + \ldots + x^{\frac{1}{n}} + 1) &= x^{\frac{n}{n}} + x^{\frac{n-1}{n}} + \ldots + x^{\frac{2}{n}} + x^{\frac{1}{n}} \\ &\quad - x^{\frac{n-1}{n}} - x^{\frac{n-2}{n}} - \ldots - x^{\frac{1}{n}} - 1 \\ &= x - 1,\end{aligned}$$

$$(x^{\frac{1}{n}} - 1)(x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}} + \cdots + x^{\frac{1}{n}} + 1) = x - 1.$$

For $x > 1$, we have

$$x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}} + \cdots + x^{\frac{1}{n}} + 1 > n, \text{ so that}$$

$$0 < x^{\frac{1}{n}} - 1 = \frac{x-1}{x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}} + \cdots + x^{\frac{1}{n}} + 1} < \frac{x-1}{n},$$

which approaches 0 as n approaches ∞.

$x^{\frac{1}{n}} - 1$ also approaches 0 .

If x equals 1, then $x^{\frac{1}{n}}$, as well as its limit, also equals 1.

If x lies between 0 and 1, then $\dfrac{1}{x}$ is greater than 1, so that we finally have

$$\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{\frac{1}{x}}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\left(\frac{1}{x}\right)^{\frac{1}{n}}} = \frac{1}{\lim\limits_{n\to\infty}\left(\dfrac{1}{x}\right)^{\frac{1}{n}}} = \frac{1}{1} = 1.$$

Later on, there will be easier proofs of

$$\boxed{\text{For } -1 < x < 1,\ \lim_{n\to\infty} x^n = 0}$$

and of

$$\boxed{\text{For } 0 < x,\ \lim_{n\to\infty} x^{\frac{1}{n}} = 1}$$

with continuity of the exponential and logarithm functions:

$$\text{For } 0 < x < 1,\ \lim_{n\to\infty} x^n = \lim_{n\to\infty} e^{n\ln x} = \lim_{y\to-\infty} e^y = 0,$$

$$\text{For } 0 < x,\ \lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{\ln x}{n}} = \lim_{y\to 0} e^y = e^0 = 1.$$