

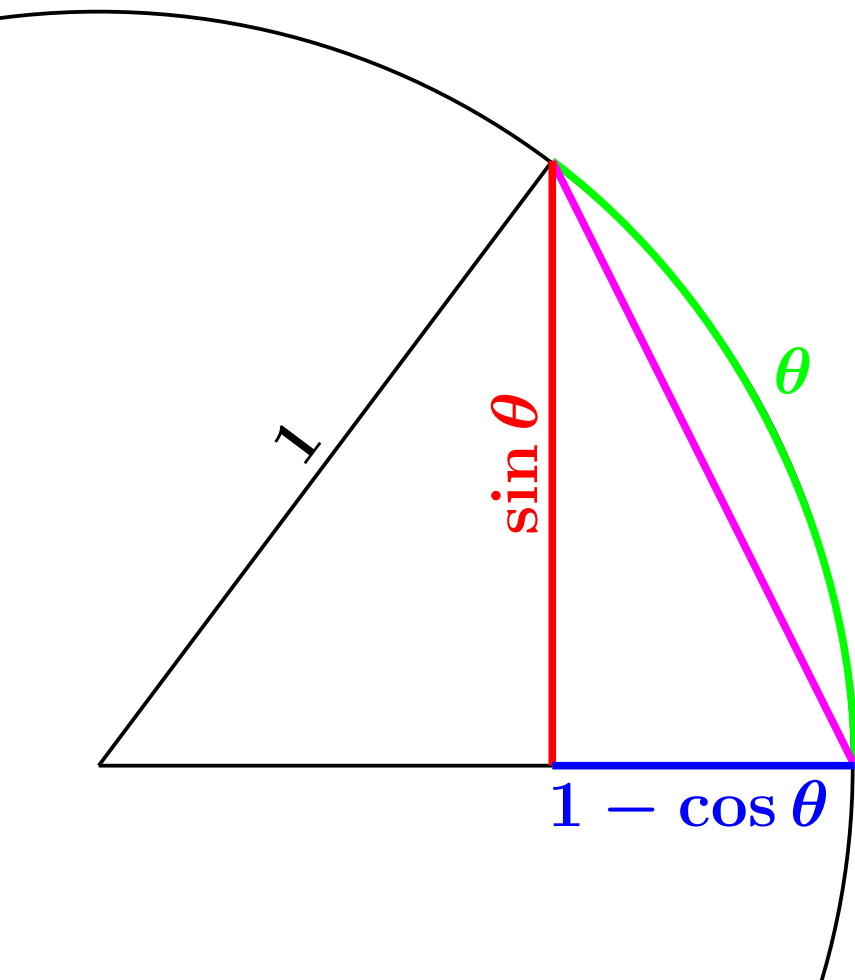
$$x \text{ (in radians)} \quad \sin x = x - \frac{x^3}{6} + o(x^4)$$

1.0	.84147098480789650665
0.1	.099833416646828152307
0.01	.0099998333341666646825
0.001	.0009999983333334166667
0.0001	.000099999983333333417
0.00001	.000009999999833333333
0.000001	.0000009999999983333333
0.0000001	.00000009999999998333333
0.00000001	.00000000999999999983333
0.000000001	.000000000999999999983

On the Continuity of Sine and Cosine at $x = 0$

For $0 \leq \theta \leq 90^\circ$, we have

$$0 \leq \sin \theta \leq \sqrt{(\sin \theta)^2 + (1 - \cos \theta)^2} \leq \theta,$$
$$0 \leq 1 - \cos \theta \leq \sqrt{(\sin \theta)^2 + (1 - \cos \theta)^2} \leq \theta,$$



As θ approaches $0+$, we get

$$\lim_{\theta \rightarrow 0^+} \sin \theta = 0,$$

$$\lim_{\theta \rightarrow 0^+} 1 - \cos \theta = 0.$$

The left limits are similar.

On the Continuity of Sine and Cosine for all x

Having gotten $\sin o(1) = o(1)$ and $\cos o(1) = 1 + o(1)$,
we can have

$$\begin{aligned}\sin(x + o(1)) &= \sin x \cos o(1) + \cos x \sin o(1) \\ &= \sin x(1 + o(1)) + \cos x \cdot o(1) \\ &= \sin x + o(1) + o(1) \\ &= \sin x + o(1), \text{ and}\end{aligned}$$

$$\begin{aligned}\cos(x + o(1)) &= \cos x \cos o(1) - \sin x \sin o(1) \\ &= \cos x(1 + o(1)) - \sin x \cdot o(1) \\ &= \cos x + o(1) + o(1) \\ &= \cos x + o(1).\end{aligned}$$

$$x \text{ (in radians)} \quad \frac{\sin x}{x} = 1 - \frac{x^2}{6} + o(x^3)$$

1.0 .84147098480789650665

0.1 .99833416646828152307

0.01 .99998333341666646825

0.001 .99999983333334166667

0.0001 .9999999983333333417

0.00001 .9999999999833333333

0.000001 .9999999999998333333

0.0000001 .9999999999999983333

0.00000001 .99999999999999998333

0.000000001 .99999999999999999983

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$$

is equivalent to

$$\frac{\sin h}{h} = 1 + o(1),$$

$$\begin{aligned}\sin h &= h(1 + o(1)) \\ &= h + h \cdot o(1)\end{aligned}$$

$$\sin h = h + o(h), \quad \text{as } h \rightarrow 0$$

Since $\cos 2u$ equals $1 - 2 \sin^2 u$, the substitution $u = \frac{h}{2}$ gives

$$\begin{aligned}\cos h &= \cos 2\frac{h}{2} = 1 - 2 \sin^2 \frac{h}{2} \\ &= 1 - 2 \left(\frac{h}{2} + o(h) \right)^2 \\ &= 1 - 2 \left(\frac{h^2}{4} + 2\frac{h}{2}o(h) + (o(h))^2 \right) \\ &= 1 - 2 \left(\frac{h^2}{4} + o(h^2) \right),\end{aligned}$$

so that

$$\begin{aligned}\cos h &= 1 - \frac{h^2}{2} + o(h^2), \\ &= 1 + o(h), \quad \text{as } h \rightarrow 0\end{aligned}$$

$$\cos h = 1 - \frac{h^2}{2} + o(h^2), \quad \text{as } h \rightarrow 0$$

is equivalent to

$$1 - \cos h = \frac{h^2}{2} + o(h^2)$$
$$\frac{1 - \cos h}{h^2} = \frac{1}{2} + o(1)$$

$$\lim_{h \rightarrow 0} \frac{1 - \cos h}{h^2} = \frac{1}{2}$$

$$\cos h = 1 + o(h), \quad \text{as } h \rightarrow 0$$

is equivalent to

$$1 - \cos h = o(h)$$
$$\frac{1 - \cos h}{h} = o(1)$$

$$\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0$$

The traditional way to get this limit:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} &= \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \cdot \frac{1 + \cos h}{1 + \cos h} \\ &= \lim_{h \rightarrow 0} \frac{1 - \cos^2 h}{h(1 + \cos h)} \\ &= \lim_{h \rightarrow 0} \frac{\sin^2 h}{h(1 + \cos h)} \\ &= \lim_{h \rightarrow 0} \frac{\sin h}{h} \cdot \frac{\sin h}{(1 + \cos h)} \\ &= 1 \cdot \frac{0}{1 + 1} \\ \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} &= 0,\end{aligned}$$

The traditional way to get the previous limit:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{1 - \cos h}{h^2} &= \lim_{h \rightarrow 0} \frac{1 - \cos h}{h^2} \cdot \frac{1 + \cos h}{1 + \cos h} \\ &= \lim_{h \rightarrow 0} \frac{1 - \cos^2 h}{h^2(1 + \cos h)} \\ &= \lim_{h \rightarrow 0} \frac{\sin^2 h}{h^2(1 + \cos h)} \\ &= \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right)^2 \cdot \frac{1}{(1 + \cos h)} \\ &= 1^2 \cdot \frac{1}{1 + 1} \\ \lim_{h \rightarrow 0} \frac{1 - \cos h}{h^2} &= \frac{1}{2},\end{aligned}$$

Using $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$, we can have

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{x} = \lim_{x \rightarrow 0} \frac{5 \sin 5x}{5x} = 5 \lim_{u \rightarrow 0} \frac{\sin u}{u} = 5 \cdot 1 = 5.$$

WARNING! $\lim_{x \rightarrow 0} \frac{\sin 5x}{x} = \lim_{x \rightarrow 0} \frac{5 \sin x}{x} = 5 \cdot 1 = 5.$

would be wrong, because $\sin 5x \neq 5 \sin x!$

(Actually, $\sin 5x$ equals $5 \sin x - 20 \sin^3 x + 16 \sin^5 x$.)

Another Example:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan 3x}{2x} &= \lim_{x \rightarrow 0} \frac{\sin 3x}{2x \cos 3x} = \lim_{x \rightarrow 0} \frac{3}{2} \frac{\sin 3x}{3x} \frac{1}{\cos 3x} \\ &= \frac{3}{2} \cdot 1 \cdot 1 = \frac{3}{2}. \end{aligned}$$