Higher Order Derivatives

$$\begin{aligned} f(x) &= 2x^5 - 6x^4 + 2x - 1 \\ \frac{d}{dx}f(x) = f'(x) &= 10x^4 - 24x^3 + 2 \\ \frac{d^2}{dx^2}f(x) = f''(x) &= 40x^3 - 72x^2 \end{aligned}$$

Notation:

$$\begin{aligned} y &= f(x) \\ \frac{d}{dx}y = \frac{dy}{dx} &= f'(x) \\ \frac{d^2}{dx^2}y = \frac{d^2y}{dx^2} &= f''(x) \\ \frac{d^3}{dx^3}y = \frac{d^3y}{dx^3} &= f^{(3)}(x) \end{aligned}$$

Example:

$$\begin{aligned}
y &= x^n \\
\frac{dy}{dx} &= nx^{n-1} \\
\frac{d^2y}{dx^2} &= n(n-1)x^{n-2} \\
\frac{d^3y}{dx^3} &= n(n-1)(n-2)x^{n-3} \\
\frac{d^4y}{dx^4} &= n(n-1)(n-2)(n-3)x^{n-4} \\
\frac{d^ky}{dx^k} &= n(n-1)(n-2)\ldots(n-k+1)x^{n-k} \\
&= \frac{n!}{(n-k)!}x^{n-k} \text{ for } k \leq n, \text{ for integer } n.
\end{aligned}$$

Example:

$$
\begin{aligned}
y &= \frac{x^n}{n!}, && \text{for integer } n \geq 0, \\
\frac{dy}{dx} &= \frac{n x^{n-1}}{n!} &&= \frac{x^{n-1}}{(n-1)!} \\
\frac{d^2y}{dx^2} &= \frac{(n-1)x^{n-2}}{(n-1)!} &&= \frac{x^{n-2}}{(n-2)!} \\
\frac{d^3y}{dx^3} &= \frac{(n-2)x^{n-3}}{(n-2)!} &&= \frac{x^{n-3}}{(n-3)!} \\
\frac{d^4y}{dx^4} &= \frac{(n-3)x^{n-4}}{(n-3)!} &&= \frac{x^{n-4}}{(n-4)!} \\
\frac{d^5y}{dx^5} &= \frac{(n-4)x^{n-5}}{(n-4)!} &&= \frac{x^{n-5}}{(n-5)!} \\
\frac{d^ky}{dx^k} & &&= \frac{x^{n-k}}{(n-k)!} \text{ for } k \leq n.
\end{aligned}
$$

Example:

$$
\begin{aligned}
y &= x^{-1}, & &= \frac{1}{x} \\
\frac{dy}{dx} &= (-1)x^{-2} & &= -\frac{1}{x^2} \\
\frac{d^2y}{dx^2} &= (-2)(-1)x^{-3} & &= \frac{2!}{x^3} \\
\frac{d^3y}{dx^3} &= (-3)(-2)(-1)x^{-4} & &= -\frac{3!}{x^4} \\
\frac{d^4y}{dx^4} &= (-4)(-3)(-2)(-1)x^{-5} & &= \frac{4!}{x^5} \\
\frac{d^5y}{dx^5} &= (-5)(-4)(-3)(-2)(-1)x^{-6} & &= -\frac{5!}{x^6} \\
\frac{d^ky}{dx^k} & & &= (-1)^k\frac{k!}{x^{k+1}}.
\end{aligned}
$$

$$\boldsymbol{f(x)g(x) = \binom{0}{0} f^{(0)}(x)g^{(0)}(x)}$$

$$\boldsymbol{\Big(f(x)g(x)\Big)' = f'(x)g(x) + f(x)g'(x)}$$

$$\boldsymbol{= \binom{1}{0} f^{(1)}(x)g^{(0)}(x) + \binom{1}{1} f^{(0)}(x)g^{(1)}(x)}$$

$$\boldsymbol{\Big(f(x)g(x)\Big)^{(2)} = f^{(2)}(x)g(x) + 2f'(x)g'(x) + f(x)g^{(2)}(x)}$$

$$\boldsymbol{= \binom{2}{0} f^{(2)}(x)g^{(0)}(x) + \binom{2}{1} f^{(1)}(x)g^{(1)}(x)}$$

$$\boldsymbol{+ \binom{2}{2} f^{(0)}(x)g^{(2)}(x)}$$

$$
\begin{aligned}
\Big(f(x)g(x)\Big)^{(3)} &= f^{(3)}(x)g(x) \;+\; 3f^{(2)}(x)g'(x) \\
&\qquad + \; 3f'(x)g^{(2)}(x) \;+\; f(x)g^{(3)}(x) \\
&= \binom{3}{0} f^{(3)}(x)g^{(0)}(x) \;+\; \binom{3}{1} f^{(2)}(x)g^{(1)}(x) \\
&\quad + \; \binom{3}{2} f^{(1)}(x)g^{(2)}(x) + \binom{3}{3} f^{(0)}(x)g^{(3)}(x)
\end{aligned}
$$

Leibnitz' Formula:

$$
\begin{aligned}
\Big(f(x)g(x)\Big)^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x) \\
&= \sum_{\text{all } k} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x)
\end{aligned}
$$

To complete the proof for higher n's, assume that

$$\begin{aligned}
\Big(f(x)g(x)\Big)^{(n)} &= \sum_{\text{all } k} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \text{ for any } n: \\
\Big(f(x)g(x)\Big)^{(n+1)} &= \frac{d}{dx} \sum_{\text{all } k} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \\
&= \sum_{\text{all } k} \binom{n}{k} f^{(k+1)}(x) g^{(n-k)}(x) \\
&\qquad + \sum_{\text{all } k} \binom{n}{k} f^{(k)} g^{(n-k+1)}(x) \\
&= \sum_{\text{all } j} \binom{n}{j} f^{(j+1)}(x) g^{(n-j)}(x) \\
&\qquad + \sum_{\text{all } j} \binom{n}{j+1} f^{(j+1)}(x) g^{(n-j)}(x)
\end{aligned}$$

substituting $k = j$ in $\uparrow$ and, in $\uparrow$, $k = j + 1,\ \ k - 1 = j$,

Assuming

$$\Big(f(x)g(x)\Big)^{(n)} = \sum_{\text{all } k} \binom{n}{k} f^{(k)} g^{(n-k)}(x) \text{ for any } n\text{, we have :}$$

$$\begin{aligned}
\Big(f(x)g(x)\Big)^{(n+1)} &= \sum_{\text{all } j} \left(\binom{n}{j} + \binom{n}{j+1}\right) f^{(j+1)}(x) g^{(n-j)}(x) \\
&= \sum_{\text{all } j} \binom{n+1}{j+1} f^{(j+1)}(x) g^{(n-j)}(x) \\
&= \sum_{\text{all } j} \binom{n+1}{j+1} f^{(j+1)}(x) g^{((n+1)-(j+1))}(x)
\end{aligned}$$

If we let $j+1$ equal k, we now have

$$\Big(f(x)g(x)\Big)^{(n+1)} = \sum_{k=0}^{n} \binom{n+1}{k} f^{(k)}(x) g^{((n+1)-k)}(x).$$