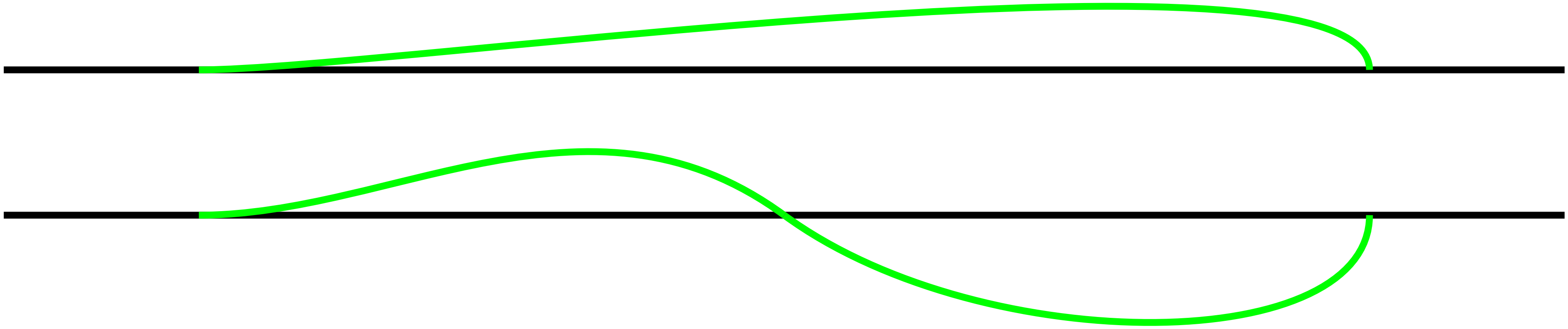
Which of these results would hold if the function f had only a one-sided derivative $f'(x+)$ or $f'(x-)$? The Inequalities.

Altering Rolle's Theorem to apply to right-hand derivatives, consider now a function $f(x)$ continuous for all x in $a \leq x \leq b$, satisfying $f(a) = f(b) = 0$, and which is right-hand differentiable for all x in $\boxed{a \leq x} < b$.



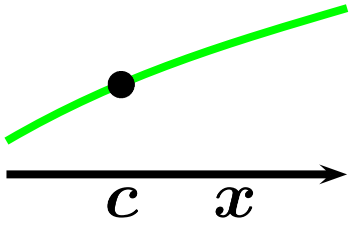
We shall show that $f'(c+) \leq 0$ for at least one c in $\boxed{a \leq x} < b$.

$f(x)$ either takes on a positive value in the interval $a \leq x \leq b$, or it doesn't.

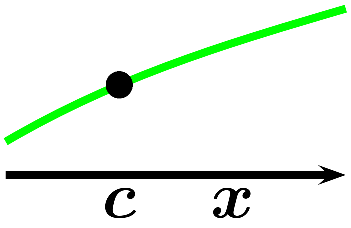
If $f(x)$ were positive for at least one x between a and b, then f's maximum value, $f(c)$ say, would also be positive.

Let c be the location of any such maximum value, $f(c)$, where $a \leq c \leq b$.
Since we would have $f(c) > 0$, and since $f(a) = f(b) = 0$,
then c would not equal either a or b,
so c would satisfy $a < c < b$.

If $f'(c+)$ were positive, a contradiction would result to the maximality of $f(c)$:



$f'(c+) > 0$ would give a contradiction:



Let $\epsilon = \dfrac{f'(c+)}{2} > 0$. A $\delta > 0$ would exist such that $c < x < c + \delta < b$ would imply that

$$\begin{aligned}
\left|\frac{f(x)-f(c)}{x-c} - f'(c+)\right| &< \frac{f'(c+)}{2} = \epsilon, \\
-\frac{f'(c+)}{2} < \frac{f(x)-f(c)}{x-c} - f'(c+) &< \frac{f'(c+)}{2}, \\
0 < \frac{f'(c+)}{2} &< \frac{f(x)-f(c)}{x-c}, \\
0 &< f(x)-f(c), \\
f(c) &< f(x),
\end{aligned}$$

which would contradict the maximality of our maximal $f(c)$.

$f'(c+)$ could not be positive, and we would have $\boxed{f'(c+) \le 0.}$

Otherwise, we would have $f(x) \leq 0$ for all x in $a \leq x \leq b$, and this would give us

$$\text{(for } a < x \leq b) \quad \frac{f(x) - f(a)}{x - a} = \frac{f(x) - 0}{x - a} = \frac{f(x)}{x - a} \leq 0,$$

$$f'(a+) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a} \leq 0.$$

In either case we know, so far, that

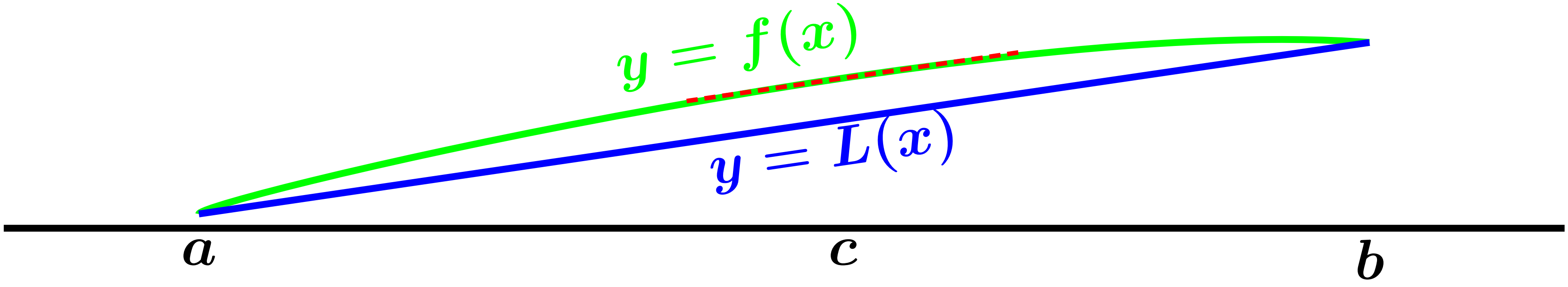
for a function $f(x)$ continuous for all x in $a \leq x \leq b$,

which satisfies $f(a) = f(b) = 0$,

and which is right-hand differentiable for all x in $\boxed{a \leq x} < b$,

that there would be a c in $\boxed{a \leq c} < b$, where $\boxed{f'(c+) \leq 0.}$

Now, consider any $f(x)$ continuous for all x in $a \le x \le b$ and right-differentiable for all x in $\boxed{a \le x} < b$, but with any values of $f(a)$ and $f(b)$:



Take a linear function L where $L(a) = f(a)$ and $L(b) = f(b)$.

This $L(x)$ would equal $\dfrac{f(b)(x-a) + f(a)(b-x)}{b-a}$.

Since $f(a) - L(a) = 0$ and $f(b) - L(b) = 0$, then the last result can be applied to the right-differentable function $f - L$,

to get $f'(c+) \;-\; L'(c+) \;=\; (f-L)'(c+) \;\le\; 0$,

and $\quad f'(c+) \;\le\; L'(c) \;=\; \dfrac{f(b) - f(a)}{b-a}$.

For a function $f(x)$ continuous for all x in $a \leq x \leq b$,
and which is right-hand differentiable for all x in $\boxed{a \leq x} < b$,
we have $\boxed{f'(c+) \leq \dfrac{f(b) - f(a)}{b - a}}$ for some c in $\boxed{a \leq c} < b$.
This is analogous to the Mean Value Theorem.

Preliminary Lemmas for Right-Hand Derivative Inequalities

If $M \leq f'(x+)$, for all x in $\boxed{a \leq x} < b$,
then f must also satisfy $M \;\; \leq \;\; f'(c+) \;\; \leq \;\; \dfrac{f(b) - f(a)}{b - a}$.

If $M < f'(x+)$, for all x in $\boxed{a \leq x} < b$,
then f must also satisfy $M \;\; < \;\; f'(c+) \;\; \leq \;\; \dfrac{f(b) - f(a)}{b - a}$.

For flexiblity the conditions $\boxed{a \leq x} < b$ will become $\boxed{a < x} < b$.

Consider a function $f(x)$ continuous for all x in $a \leq x \leq b$, and satisfying $M \leq f'(x+)$ only for the x's in $\boxed{a < x} < b$.

The $\leq$-part of the Lemma is still satisfied on the interval $a < a + h \leq x \leq b$, and it yields $M \leq \dfrac{f(b) - f(a+h)}{b - (a+h)}$. Since f is continuous, we can take a limit as h approaches 0 and we will have $M \leq \dfrac{f(b) - f(a)}{b - a}$, as desired.

If $M \leq f'(x+)$, for all x in $\boxed{a < x} < b$,

then f must also satisfy $M \ \leq \ \dfrac{f(b) - f(a)}{b - a}$.

If $M < f'(x+)$, for all x in $\boxed{a \leq x} < b$,

then f must also satisfy $M \ < \ \dfrac{f(b) - f(a)}{b - a}$.

Consider a function $f(x)$ continuous for all x in $a \leq x \leq b$, and satisfying $M < f'(x+)$ only for the x's in $\boxed{a < x} < b$.

Trivially, $M \leq f'(x+)$ is also satisfied, there.
The improved $\leq$-part of the Lemma holds on the interval $a < x < \dfrac{a+b}{2}$,

$$\text{and it yields } M \leq \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}.$$

The original $<$-part of the Lemma holds on the interval $\dfrac{a+b}{2} \leq x < b$,

$$\text{and it yields } M < \frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}}.$$

We have $$M\left(b - \frac{a+b}{2}\right) < f(b) - f\left(\frac{a+b}{2}\right).$$

and $$M\left(\frac{a+b}{2} - a\right) \le f\left(\frac{a+b}{2}\right) - f(a).$$

If we add, we have $$M(b-a) < f(b) - f(a),$$

so that $$M < \frac{f(b) - f(a)}{b-a}.$$

If $M \le f'(x+)$, **for all** x **in** $\boxed{a < x} < b$,

then f **must also satisfy** $M \le \dfrac{f(b) - f(a)}{b-a}$.

If $M < f'(x+)$, **for all** x **in** $\boxed{a < x} < b$,

then f **must also satisfy** $M < \dfrac{f(b) - f(a)}{b-a}$.

These Lemmas no longer depend on $f'(a+)$.

Similarly,
if a function $f(x)$ is continuous for all x in $a \leq x \leq b$,
and if f satisfies $f'(x+) \leq M$ or $f'(x+) < M$.
for all x in $\boxed{a < x} < b$,
then it also satisfies $-f'(x+) \geq M$ or $-f'(x+) < M$,
and these Lemmas can also be applied to $-f$.

Inequality Corollary/Version of the Mean Value Theorem for Right-Hand Derivatives:

If a function $f(x)$ is continuous for all x in $a \leq x \leq b$,

if $f(x)$ is right-hand differentiable

for all x in $a < x < b$,

and if $f'(x+)$ is $< M, \ \leq M, \ = M, \ \geq M, \ > M,$

respectively, for all x in $a < x < b$,

then $$\frac{f(b) - f(a)}{b - a}$$

is also $< M, \ \leq M, \ = M, \ \geq M, \ > M,$

respectively.

By Right-Left Symmetry, we also have:

Inequality Corollary/Version of the Mean Value Theorem for Left-Hand Derivatives:

If a function $f(x)$ is continuous for all x in $a \le x \le b$,

if $f(x)$ is left-hand differentiable

for all x in $a < x < b$,

and if $f'(x-)$ is $< M,\ \le M,\ = M,\ \ge M,\ > M,$

respectively, for all x in $a < x < b$,

then $\dfrac{f(b) - f(a)}{b - a}$

is also $< M,\ \le M,\ = M,\ \ge M,\ > M,$

respectively.