

The Lagrange Mean Value Theorem for Derivatives and the Intermediate Value Theorem for Continuous Functions yield Darboux' Intermediate Value Theorem for Derivatives:

If $f'(x)$ exists for $a \leq x \leq b$, with $f'(a) \neq f'(b)$ then for any number d between $f'(a)$ and $f'(b)$ there is a number c in $a < c < b$ where $f'(c) = d$.

Proof:

If d equals $\frac{f(b) - f(a)}{b - a}$,

the Mean Value Theorem gives a number c

where $f'(c) = \frac{f(b) - f(a)}{b - a} = d$.

For the rest, assume that d is not equal to $\frac{f(b) - f(a)}{b - a}$.

There are six possible orderings of $f'(a)$, $f'(b)$ and $\frac{f(b) - f(a)}{b - a}$:

$$f'(a) < f'(b) \leq \frac{f(b) - f(a)}{b - a},$$

$$f'(a) \leq \frac{f(b) - f(a)}{b - a} \leq f'(b),$$

$$\frac{f(b) - f(a)}{b - a} \leq f'(a) < f'(b),$$

$$f'(b) < f'(a) \leq \frac{f(b) - f(a)}{b - a},$$

$$f'(b) \leq \frac{f(b) - f(a)}{b - a} \leq f'(a),$$

$$\frac{f(b) - f(a)}{b - a} \leq f'(b) < f'(a).$$

With $d \neq f'(a)$, $d \neq f'(b)$ and $d \neq \frac{f(b) - f(a)}{b - a}$, we have

$$f'(a) < d < f'(b) \leq \frac{f(b) - f(a)}{b - a},$$

$$f'(a) < d < \frac{f(b) - f(a)}{b - a} \leq f'(b),$$

$$f'(a) \leq \frac{f(b) - f(a)}{b - a} < d < f'(b),$$

$$\frac{f(b) - f(a)}{b - a} \leq f'(a) < d < f'(b),$$

$$f'(b) < d < f'(a) \leq \frac{f(b) - f(a)}{b - a},$$

$$f'(b) < d < \frac{f(b) - f(a)}{b - a} \leq f'(a),$$

$$f'(b) \leq \frac{f(b) - f(a)}{b - a} < d < f'(a),$$

$$\frac{f(b) - f(a)}{b - a} \leq f'(b) < d < f'(a).$$

In each of the possible cases, above,

we have either

that d lies between $f'(a)$ and $\frac{f(b) - f(a)}{b - a}$,

or

that d lies between $f'(b)$ and $\frac{f(b) - f(a)}{b - a}$.

If d lies between $f'(a)$ and $\frac{f(b) - f(a)}{b - a}$,

define $g(x)$ to be $\begin{cases} \frac{f(x) - f(a)}{x - a} & \text{for } x \neq a, \\ f'(a) & \text{for } x = a. \end{cases}$ (continuous)

d thus lies between $g(a) = f'(a)$ and $g(b) = \frac{f(b) - f(a)}{b - a}$.

The Intermediate Value Theorem applied to $g(x)$ yields an x_1 between a and b where $g(x_1) = \frac{f(x_1) - f(a)}{x_1 - a}$ equals d .

The Mean Value Theorem applied to $f(x)$ between a and x_1 yields a c such that $f'(c) = \frac{f(x_1) - f(a)}{x_1 - a}$ also equals d .

If d lies between $f'(b)$ and $\frac{f(b) - f(a)}{b - a}$,

define $h(x)$ to be $\begin{cases} \frac{f(x) - f(b)}{x - b} & \text{for } x \neq b, \\ f'(b) & \text{for } x = b. \end{cases}$ (continuous)

d thus lies between $h(b) = f'(b)$ and $h(a) = \frac{f(a) - f(b)}{a - b}$.

The Intermediate Value Theorem applied to $h(x)$ yields an x_1 between a and b where $h(x_1) = \frac{f(x_1) - f(b)}{x_1 - b}$ equals d .

The Mean Value Theorem applied to $f(x)$ between b and x_1 yields a c such that $f'(c) = \frac{f(x_1) - f(b)}{x_1 - b}$ also equals d .

Intermediate Value Theorem for Derivatives:

If $f'(x)$ exists for $a \leq x \leq b$, with $f'(a) \neq f'(b)$
then for any number d between $f'(a)$ and $f'(b)$
there is a number c in $a < c < b$ where $f'(c) = d$.

Application:

If $f'(x)$ exists, with $f'(x) \neq 0$, on any interval
then f has a differentiable inverse, there.

Proof:

$f'(x)$ could not change sign without taking the value 0.

$f'(x)$ is either always positive or always negative.

$f(x)$ is either strictly increasing or strictly decreasing.

$f(x)$ is one-to one. Thus f has an inverse f^{-1} .

Since f is differentiable, so is f^{-1} .

Intermediate Value Theorem for Derivatives:

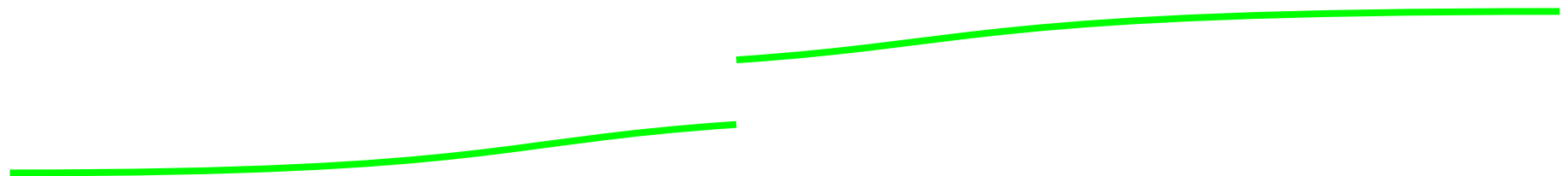
If $f'(x)$ exists for $a \leq x \leq b$, with $f'(a) \neq f'(b)$
then for any number d between $f'(a)$ and $f'(b)$
there is a number c in $a < c < b$ where $f'(c) = d$.

Application:

If $f'(x)$ is increasing or decreasing on an interval,
then f' is continuous there, so f is smooth there.

Proof:

A monotone $f'(x)$ which is not continuous would have a jump,



contradicting the Derivative Intermediate Value Theorem.