Iteration

To solve $\phi(x) = x$ for x, start with some x_0 nearby, and

$$\begin{aligned} \textbf{let } x_1 &\ \textbf{equal}\ \ \phi(x_0), \\ x_2 &\ \textbf{equal}\ \ \phi(x_1), \\ x_3 &\ \textbf{equal}\ \ \phi(x_2), \\ &\ \ \cdot \\ &\ \ \cdot \\ x_{n+1} &\ \textbf{equal}\ \ \phi(x_n), \\ &\ \ \cdot \\ &\ \ \cdot \\ \textbf{Then } x &\ \textbf{equals}\ \ \phi(x), \end{aligned}$$

if the sequence of x_n's approaches x,
and if the function ϕ is continuous at x.

Example: To solve $x = \sqrt[4]{x+14}$, let x_0 equal 0, and

$$\begin{aligned}
x_1 &= \sqrt[4]{0+14} = 1.9979448, \\
x_2 &= \sqrt[4]{1.9979448+14} = 1.9999357, \\
x_3 &= \sqrt[4]{1.9999357+14} = 1.9999979, \\
x_4 &= \sqrt[4]{1.9999979+14} = 1.9999998, \\
&\quad\vdots \\
x &= \sqrt[4]{1.9999998+14} = 1.9999999 \\
&\cong 2.0000000
\end{aligned}$$

Note that $\sqrt[4]{2+14} = \sqrt[4]{16} = 2$.

Also, to solve $x = \sqrt[4]{x+14}$, two more iterations:

$$
\begin{aligned}
x_0 &= 100.0000000 & \qquad x_0 &= 1,000,000.0000000 \\
x_1 &= 3.2675798 & \qquad x_1 &= 31.622885 \\
x_2 &= 2.0384865 & \qquad x_2 &= 2.5989365 \\
x_3 &= 2.0012015 & \qquad x_3 &= 2.0184595 \\
x_4 &= 2.0000375 & \qquad x_4 &= 2.0005765 \\
x_5 &= 2.0000011 & \qquad x_5 &= 2.0000175 \\
x_6 &= 2.0000000 & \qquad x_6 &= 2.0000005
\end{aligned}
$$

Since $\phi(x)$ equals $\sqrt[4]{x+14} \;=\; (x+14)^{\frac{1}{4}}$, and since $\phi'(x)$ equals $\frac{1}{4}(x+14)^{-\frac{3}{4}} \le \frac{1}{4}(0+14)^{-\frac{3}{4}} \le \frac{1}{4}(14)^{-\frac{3}{4}} \le \frac{1}{28}$, for $x \ge 0$, the Mean Value Theorem gives us

$$|x_{n+1} - 2| \;=\; |\phi(x_n) - \phi(2)| \;\le\; \frac{1}{28}\,|x_n - 2|$$

Whenever the x_n and their limit x are in an interval where $|\phi'(x)| \le M < 1$, then we have

$$
\begin{aligned}
-M &\le \phi'(x) \le M \\
-M &\le \frac{\phi(x_n) - \phi(x)}{x_n - x} \le M \\
-M &\le \frac{x_{n+1} - x}{x_n - x} \le M \\
&\left|\frac{x_{n+1} - x}{x_n - x}\right| \le M \\
&\left|x_{n+1} - x\right| \le M \cdot \left|x_n - x\right|,
\end{aligned}
$$

so that each x_n is closer than its predecessor by a factor at least as small as M.

$$\begin{aligned}
\text{Since we have } \left|x_n - x\right| &\leq M \cdot \left|x_{n-1} - x\right| \\
&\leq M \cdot M \cdot \left|x_{n-2} - x\right| \\
&\leq M \cdot M \cdot M \cdot \left|x_{n-3} - x\right| \\
&\leq \ldots, \\
\text{we finally have } \left|x_n - x\right| &\leq M^n \cdot \left|x_0 - x\right|,
\end{aligned}$$

which approaches 0 as $n \longrightarrow \infty$.

Thus the x_n approach x.

On the other hand, if $|\phi'(x)| \geq 1$, the $x_n's$ get farther and farther apart, and thus they diverge.

For $x < 0$, $x^4 - x - 14 = 0$ also means that $x = -\sqrt[4]{x+14}$:

$$
\begin{aligned}
x_0 &= 0.000000000000000 \\
x_1 &= -1.93433642026767 \\
x_2 &= -1.86375062881515 \\
x_3 &= -1.86647046865692 \\
x_4 &= -1.86636588677009 \\
x_5 &= -1.86636990842419 \\
x_6 &= -1.86636975377359 \\
x_7 &= -1.86636975972060 \\
x_8 &= -1.86636975950070
\end{aligned}
$$

Since $\phi(x)$ equals $-\sqrt[4]{x+14} \;=\; -(x+14)^{\frac{1}{4}}$, and since $|\phi'(x)|$ equals $|-\frac{1}{4}(x+14)^{-\frac{3}{4}}| \leq \frac{1}{4}(-2+14)^{-\frac{3}{4}} \leq 0.04 = \frac{1}{25}$, for $x \leq 0$, the MVT gives $|x_{n+1} - 2| \;\leq\; \frac{1}{25}\;|x_n - 2|$.

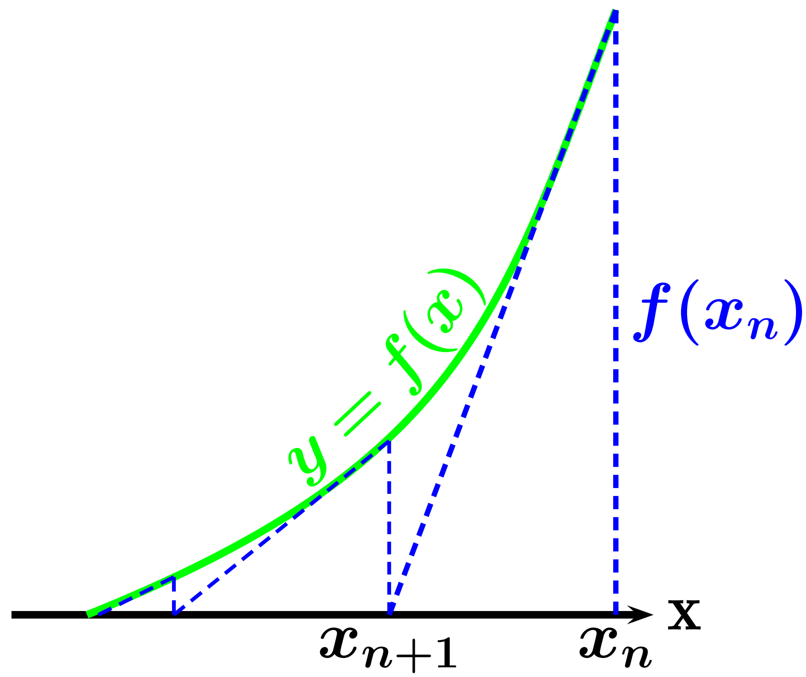
Newton's Method–For Approximating Solutions to $f(x) = 0$:

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

$$x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



$$x_{n+1} = \phi(x_n), \qquad \text{where } \phi(u) = u - \frac{f(u)}{f'(u)}.$$

(Note that $\phi(x) = x$ iff $f(x) = 0$.)

To determine convergence, we check to see if $|\phi'(x)| < 1$:

$$\begin{aligned}
\phi'(x) &= \frac{d}{dx}\left(x - \frac{f(x)}{f'(x)}\right) \\
&= 1 - \frac{f'(x) \cdot f'(x) - f(x) \cdot f''(x)}{(f'(x))^2} \\
&= \frac{f'(x) \cdot f'(x)}{(f'(x))^2} - \frac{f'(x) \cdot f'(x) - f(x) \cdot f''(x)}{(f'(x))^2} \\
&= \frac{f(x) \cdot f''(x)}{(f'(x))^2}
\end{aligned}$$

If $f'(x)$ stays away from 0, and if $f''(x)$ doesn't grow much, then $\phi'(x)$ will approach 0 as $f(x)$ does, and convergence here will be much faster than it is with ordinary iteration, where $|\phi'(x)| < M \leq 1$ would have sufficed.

Using Newton's Method to solve $f(x) = x^4 - x - 14 = 0$, by iterating

$$x \longrightarrow \phi(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^4 - x - 14}{4x^3 - 1} = \frac{3x^4 + 14}{4x^3 - 1}$$

$$
\begin{aligned}
x_0 &= 100.00000000 & x_9 &= 7.516800847 \\
x_1 &= 75.00002225 & x_{10} &= 5.649166684 \\
x_2 &= 56.25005832 & x_{11} &= 4.262199457 \\
x_3 &= 42.18762266 & x_{12} &= 3.252353679 \\
x_4 &= 31.64086896 & x_{13} &= 2.559601542 \\
x_5 &= 23.73094950 & x_{14} &= 2.160625768 \\
x_6 &= 17.79880697 & x_{15} &= 2.017473769 \\
x_7 &= 13.35031786 & x_{16} &= 2.000232796 \\
x_8 &= 10.01526160 & x_{17} &= 2.00000041948 \\
 & & x_{18} &= 2.0000000000000013623
\end{aligned}
$$

Using Newton's Method to solve $f(x) = x^4 - x - 14 = 0$, by iterating

$$x \longrightarrow \phi(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^4 - x - 14}{4x^3 - 1} = \frac{3x^4 + 14}{4x^3 - 1}$$

$$\begin{aligned} x_0 &= 0.000 & x_7 &= -2.5980 \\ x_1 &= -14.000 & x_8 &= -2.1179 \\ x_2 &= -10.500 & x_9 &= -1.9067 \\ x_3 &= -7.877 & x_{10} &= -1.8676 \\ x_4 &= -5.912 & x_{11} &= -1.8663709 \\ x_5 &= -4.445 & x_{12} &= -1.866369759501389 \\ x_6 &= -3.364 & x_{13} &= -1.86636975950037621146202891 \end{aligned}$$

In most examples the number of correct digits will ultimately double, roughly, with each step.

Example: To find $\sqrt{c}$:

Use Newton's Method on $f(x) = x^2 - c$:

Iterate

$$\phi(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - c}{2x} = \frac{2x^2 - (x^2 - c)}{2x}$$

$$= \frac{x^2 + c}{2x} = \frac{x + \frac{c}{x}}{2}.$$

To guarantee convergence, we make sure that

$$|\phi'(x)| = \left|\frac{(2x)(2x) - 2(x^2 + c)}{(2x)^2}\right| = \left|\frac{x^2 - c}{2x^2}\right| < 1,$$

by noting that if $x_0 \geq 0$ then every later x_k is $\geq \sqrt{c}$,
and that $|\phi'(x)|$ is smaller than $\frac{1}{2}$ for all these x.

Example: To find $\sqrt{2}$, use Newton's Method on $f(x) = x^2 - 2$.

Iterate $\phi(x) = \dfrac{x^2+2}{2x}$:

$$x_0 = \frac{1}{1} = 1.000000000000000000000000000$$

$$x_1 = \frac{3}{2} = 1.500000000000000000000000000$$

$$x_2 = \frac{17}{12} = 1.416666666666666666666666667$$

$$x_3 = \frac{577}{408} = 1.414215686274509803921568 6$$

$$x_4 = \frac{665857}{470832} = 1.414213562374689910626295 5$$

$$x_5 = \frac{886731088897}{627013566048} = 1.414213562373095048801689 6$$

Continued Fractions Approaching $\sqrt{2}$

$$\frac{3}{2} = 1 + \frac{1}{2}$$

$$\frac{17}{12} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}$$

$$\frac{577}{408} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}}}}}$$

$$\frac{665857}{470832} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}}}}}}}}}}}}}$$

Let x equal $2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$.

Then x satisfies

$$
\begin{aligned}
x &= 2 + \frac{1}{x} \\
x^2 &= 2x + 1 \\
x^2 - 2x &= 1 \\
x^2 - 2x + 1 &= 2 \\
x - 1 &= \pm\sqrt{2} \\
x - 1 &= \sqrt{2}, \text{ since } x > 1.
\end{aligned}
$$

Thus we have $\sqrt{2} = x - 1$ and we finally obtain:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}},$$