

A Clarification of Terms:

A Definite Integral,

$$\int_a^b f(x)dx = \left[F(x) \right]_a^b = F(b) - F(a),$$

has endpoints,

and it is a number.

An Indefinite Integral,

$$\int f(x)dx = F(x),$$

has no endpoints,

but it is a function

used inside $\left[\right]_a^b$ to find that number.

Synonyms for Indefinite Integral:

Integral Function

Antiderivative

(due to the Fundamental Theorem

and the Mean Value Theorem)

Primitive

(often found in French texts and clones thereof)

We now apply

the Chain Rule:

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x)$$

to the function $F(u) = \int_{t=a}^u f(t) dt$,

with derivative $F'(u) = f(u)$ wherever f is continuous,

$$\begin{aligned} \text{to get: } \frac{d}{dx} \int_{t=a}^{g(x)} f(t) dt &= \frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x) \\ &= f(g(x)) \cdot g'(x), \end{aligned}$$

and we have an **Extended Fundamental Theorem of Calculus**:

$$\frac{d}{dx} \int_{t=a}^{g(x)} f(t) dt = f(g(x))g'(x)$$

An Example:

$$\frac{d}{dx} \int_{t=0}^{x^2} \frac{1}{\sqrt{1+t^2}} dt = \frac{1}{\sqrt{1+(x^2)^2}} (2x) = \frac{2x}{\sqrt{1+x^4}}.$$

Since $\int_{t=g_1(x)}^{g_2(x)} = \int_{t=a}^{g_2(x)} - \int_{t=a}^{g_1(x)}$, we also have

$$\frac{d}{dx} \int_{t=g_1(x)}^{g_2(x)} f(t) dt = f(g_2(x))g_2'(x) - f(g_1(x))g_1'(x)$$

Another Example:

$$\begin{aligned} \frac{d}{dx} \int_{t=1-x}^{1+x} \frac{t-1}{t} dt &= \frac{(1+x)-1}{(1+x)} (1+x)' - \frac{(1-x)-1}{(1-x)} (1-x)' \\ &= \frac{x}{(1+x)} (1) - \frac{-x}{(1-x)} (-1) = \frac{-2x^2}{1-x^2} \end{aligned}$$

$$\frac{d}{dx} \int_{t=a}^{g(x)} f(t) dt = f(g(x))g'(x)$$

An Example:

$$\frac{d}{dx} \int_{t=0}^{x^2} \frac{1}{\sqrt{1+t^2}} dt = \frac{1}{\sqrt{1+(x^2)^2}}(2x) = \frac{2x}{\sqrt{1+x^4}}$$

Another Example, using $\int_{t=g_1(x)}^{g_2(x)} = \int_{t=a}^{g_2(x)} - \int_{t=a}^{g_1(x)}$,

$$\begin{aligned} & \frac{d}{dx} \int_{t=1-x}^{1+x} \frac{t-1}{t} dt \\ &= \frac{(1+x)-1}{(1+x)}(1+x)' - \frac{(1-x)-1}{(1-x)}(1-x)' \\ &= \frac{x}{(1+x)}(1) - \frac{-x}{(1-x)}(-1) = \frac{-2x^2}{1-x^2} \end{aligned}$$

$$\frac{d}{dx} \int_{t=a}^{g(x)} f(t) dt = f(g(x))g'(x)$$

Still Another Example:

$$P(x) = \int_{t=a}^x f(g(t))g'(t) dt, \quad P'(x) = f(g(x))g'(x).$$

$$Q(x) = \int_{u=g(a)}^{g(x)} f(u) du, \quad Q'(x) = f(g(x))g'(x).$$

Since $P'(x) = Q'(x)$, we have $P(x) = Q(x)$ plus a constant.

$$\text{Since } P(a) = \int_{t=a}^a f(g(t))g'(t) dt = 0,$$

$$\text{and since } Q(a) = \int_{u=g(a)}^{g(a)} f(u) du = 0, \text{ we have } P(x) = Q(x),$$

yielding, if $x = b$,

$$\int_{t=a}^b f(g(t))g'(t) dt = \int_{u=g(a)}^{g(b)} f(u) du$$

Another way to get

$$\int_{t=a}^b f(g(t))g'(t) dt = \int_{u=g(a)}^{g(b)} f(u) du$$

Let F be any continuous piecewise antiderivative of f .

Then we have both

$$\int_{t=a}^b F'(g(t))g'(t) dt = \left[F(g(t)) \right]_{t=a}^b = F(g(b)) - F(g(a))$$

$$\int_{u=g(a)}^{g(b)} F'(u) du = \left[F(u) \right]_{u=g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

Substitution in Indefinite Integrals (Antiderivatives)

In order to implement

$$\int f(g(t))g'(t) dt = \int f(u) du$$

substitute u for $g(t)$, so that $f(g(t))$ can become $f(u)$,
and, using $\frac{du}{dt} = g'(t)$, substitute du for $g'(t) dt$.

Solving for t 's in terms of u should not be necessary, unless there may be an extra t not converted to terms in u .

After integrating $\int f(u) du$, it may be desirable to resubstitute $g(t)$ for u .

$$\begin{aligned} & \int \frac{dx}{\sqrt{2x+1}} \\ &= \int (2x+1)^{-\frac{1}{2}} dx \\ &= \int u^{-\frac{1}{2}} \frac{du}{2} \\ &= \frac{1}{2} \int u^{-\frac{1}{2}} du \\ &= \frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= u^{\frac{1}{2}} + C \\ &= \sqrt{u} + C \\ &= \sqrt{2x+1} + C \end{aligned}$$

$$\text{taking } u = 2x + 1$$

$$\frac{du}{dx} = 2$$

$$du = 2dx$$

$$\frac{du}{2} = dx$$

$$\int t(1 + t^2)^3 dt$$

$$= \int (1 + t^2)^3 t dt$$

$$= \int u^3 \frac{du}{2}$$

$$= \frac{1}{2} \int u^3 du$$

$$= \frac{1}{2} \frac{u^4}{4} + C$$

$$= \frac{u^4}{8} + C$$

$$= \frac{(1 + t^2)^4}{8} + C$$

$$\text{taking } u = 1 + t^2$$

$$\frac{du}{dt} = 2t$$

$$du = 2t dt$$

$$\frac{du}{2} = t dt$$

However, with one less t in the integrand,
the problem becomes:

$$\begin{aligned} & \int (1 + t^2)^3 dt \\ &= \int 1 + 3t^2 + 3t^4 + t^6 dt \\ &= t + 3\frac{t^3}{3} + 3\frac{t^5}{5} + \frac{t^7}{7} + C \\ &= t + t^3 + 3\frac{t^5}{5} + \frac{t^7}{7} + C \end{aligned}$$

When a $\sqrt{\quad}$ occurs,

it is often convenient to let the radicand be u :

$$\int \frac{2x}{\sqrt[3]{6-5x^2}} dx \quad \text{with } u = 6 - 5x^2$$

$$= \int (6-5x^2)^{-\frac{1}{3}} 2x dx \quad \frac{du}{dx} = -10x$$

$$-\frac{du}{5} = 2x dx$$

$$= \int u^{-\frac{1}{3}} \left(-\frac{du}{5}\right)$$

$$= \left(-\frac{1}{5}\right) \int u^{-\frac{1}{3}} du$$

$$= \left(-\frac{1}{5}\right) \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C = -\frac{3}{10} (6-5x^2)^{\frac{2}{3}} + C$$

It may be necessary to solve for some x 's in terms of u :

$$\begin{aligned} & \int 2x\sqrt{x-1} \, dx \\ &= \int 2x(x-1)^{\frac{1}{2}} \, dx, && \text{with } u = x-1, \\ &= \int 2xu^{\frac{1}{2}} \, du && du = dx, \\ &= \int 2(u+1)u^{\frac{1}{2}} \, du, && \text{with } x = u+1, \\ &= \int 2u^{\frac{3}{2}} + 2u^{\frac{1}{2}} \, du \\ &= 2\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + 2\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C &&= \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + C \end{aligned}$$

Substitution in Definite Integrals

In order to implement

$$\int_{t=a}^b f(g(t))g'(t) dt = \int_{u=g(a)}^{g(b)} f(u) du$$

substitute u for $g(t)$, so that $f(g(t))$ can become $f(u)$,
and, using $\frac{du}{dt} = g'(t)$, substitute du for $g'(t) dt$.

Let endpoints $u = g(a)$ and $g(b)$ replace $t = a$ and b .

Solving for t 's in terms of u should not be necessary, unless there may be an extra t not yet converted to terms in u .

After $\int_{u=g(a)}^{g(b)} f(u) du$, has been integrated, there is probably no need to substitute back to t .

$$\int_{x=-1}^0 3x^2(4+2x^3)^2 dx \quad \text{with } u = 4+2x^3$$

$$\int_{x=-1}^0 (4+2x^3)^2 3x^2 dx \quad \frac{du}{dx} = 6x^2$$

$$\quad \quad \quad du = 6x^2 dx$$

$$\quad \quad \quad \frac{du}{2} = 3x^2 dx$$

When x equals -1 , u equals $4+2x^3 = 4+2(-1)^3 = 4-2 = 2$.

When x equals 0 , u equals $4+2x^3 = 4+2 \cdot 0^3 = 4-0 = 4$.

$$\int_{u=2}^4 u^2 \frac{du}{2} = \frac{1}{2} \int_{u=2}^4 u^2 du = \left[\frac{u^3}{6} \right]_{u=2}^4 = \frac{4^3}{6} - \frac{2^3}{6}$$

$$= \frac{64-8}{6} = \frac{56}{6} = \frac{28}{3}$$

$$\int_{x=0}^a x \sqrt{a^2 + x^2} \, dx$$

$$\text{with } u = a^2 + x^2$$

$$= \int_{x=0}^a x (a^2 + x^2)^{\frac{1}{2}} \, dx$$

$$\frac{du}{dx} = 2x$$

$$= \int_{x=0}^a (a^2 + x^2)^{\frac{1}{2}} x \, dx$$

$$du = 2x \, dx$$

$$\frac{du}{2} = x \, dx$$

$$= \int_{u=a^2}^{2a^2} u^{\frac{1}{2}} \frac{du}{2} = \int_{u=a^2}^{2a^2} \frac{1}{2} u^{\frac{1}{2}} \, du = \left[\frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_{u=a^2}^{2a^2}$$

$$= \left[\frac{u^{\frac{3}{2}}}{3} \right]_{u=a^2}^{2a^2} = \left[\frac{(2a^2)^{\frac{3}{2}}}{3} \right] - \left[\frac{(a^2)^{\frac{3}{2}}}{3} \right] = \frac{2\sqrt{2} - 1}{3} a^3$$

$$\int_{x=0}^1 \sqrt{1-x^2} \, dx$$

$$\text{with } x = \sin u$$

$$\frac{dx}{du} = \cos u$$

$$dx = \cos u \, du$$

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \cos u$$

$$= \int_{u=0}^{\frac{\pi}{2}} \cos u \cos u \, du = \int_{u=0}^{\frac{\pi}{2}} \cos^2 u \, du = \int_{u=0}^{\frac{\pi}{2}} \frac{1 + \cos 2u}{2} \, du$$

$$= \frac{1}{2} \left[u + \frac{\sin 2u}{2} \right]_{u=0}^{\frac{\pi}{2}} = \frac{1}{2} \left[\frac{\pi}{2} + \frac{\sin \pi}{2} \right] - \frac{1}{2} \left[0 + \frac{\sin 0}{2} \right] = \frac{\pi}{4}$$

A Useful Short Cut:

$$\text{If } \int f(x) \, dx = g(x) + C,$$

$$\text{then } \int f(ax + b) \, dx = \frac{1}{a} g(ax + b) + C,$$

for any constants, $a \neq 0$ and b .

Example: Since $\int x^{17} \, dx = \frac{x^{18}}{18} + C,$

$$\text{then } \int (x + 3)^{17} \, dx = \frac{(x + 3)^{18}}{18} + C,$$

$$\text{and } \int (2x - 5)^{17} \, dx = \frac{1}{2} \frac{(2x - 5)^{18}}{18} + C.$$

$$\text{If } \int f(x) \, dx = g(x) + C,$$

$$\text{then } \int f(ax + b) \, dx = \frac{1}{a} g(ax + b) + C,$$

for any constants, $a \neq 0$ and b .

Example: Since $\int \frac{1}{x^2} \, dx = -\frac{1}{x} + C,$

$$\text{then } \int \frac{1}{(x - 10)^2} \, dx = -\frac{1}{(x - 10)} + C,$$

$$\text{and } \int \frac{1}{(6x - 10)^2} \, dx = -\frac{1}{6} \frac{1}{(6x - 10)} + C.$$

$$\text{If } \int f(x) \, dx = g(x) + C,$$

$$\text{then } \int f(ax + b) \, dx = \frac{1}{a} g(ax + b) + C,$$

for any constants, $a \neq 0$ and b .

$$\text{Proof: } \int f(ax + b) \, dx,$$

$$\text{with } u = ax + b,$$

$$\frac{du}{dx} = a,$$

$$du = a \, dx,$$

$$\frac{du}{a} = dx,$$

$$= \int f(u) \frac{du}{a} = \frac{1}{a} \int f(u) du = \frac{1}{a} g(u) = \frac{1}{a} g(ax + b).$$

$$\text{If } \int f(x) \, dx = g(x) + C,$$

$$\text{then } \int f(ax + b) \, dx = \frac{1}{a} g(ax + b) + C,$$

for any constants, $a \neq 0$ and b .

Example: Since $\int \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x} + C,$

$$\text{then } \int \frac{1}{\sqrt{2x+1}} \, dx = \frac{1}{2} \cdot 2\sqrt{2x+1} + C.$$

$$= \sqrt{2x+1} + C.$$