Hyperbolic Functions (Euler)

$\dfrac{e^x + e^{-x}}{2}$ will be called the Hyperbolic Cosine, $\cosh x$.

$\dfrac{e^x - e^{-x}}{2}$ will be called the Hyperbolic Sine, $\sinh x$.

Symmetrically, sinh and cosh are derivatives,

and also antiderivatives, of each other.

The Hyperbolic Tangent is their quotient:

$$\tanh x \equiv \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

We also have

$$\text{sech } x \equiv \frac{1}{\cosh x}, \quad \text{cosech } x \equiv \frac{1}{\sinh x}, \quad \text{cotanh } x \equiv \frac{\cosh x}{\sinh x}.$$

$$\boxed{\cosh^2 x - \sinh^2 x = 1.}$$

Proof.

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)$$

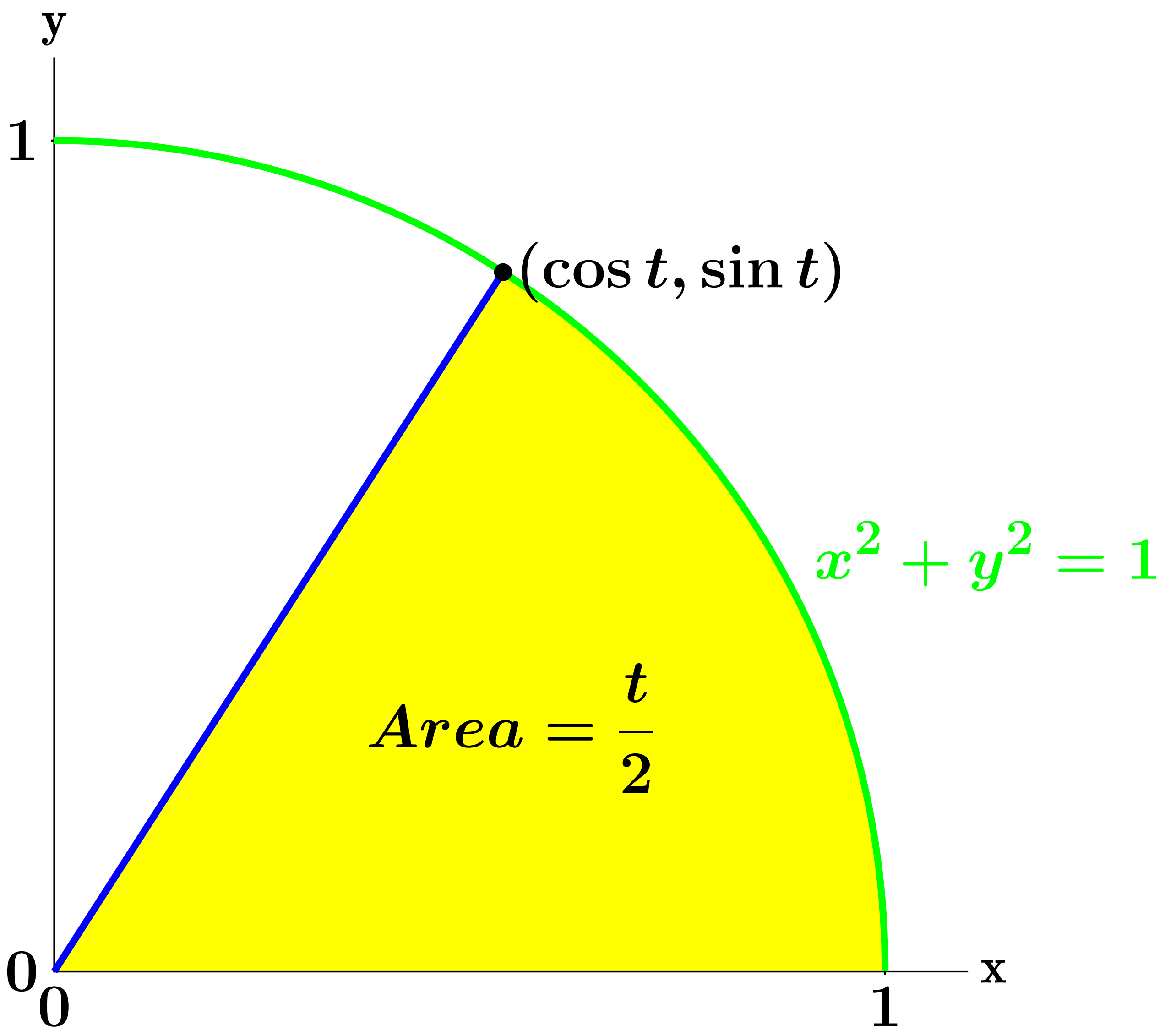
$$= \left(\frac{2\ e^x}{2}\right)\left(\frac{2\ e^{-x}}{2}\right) = 1.$$

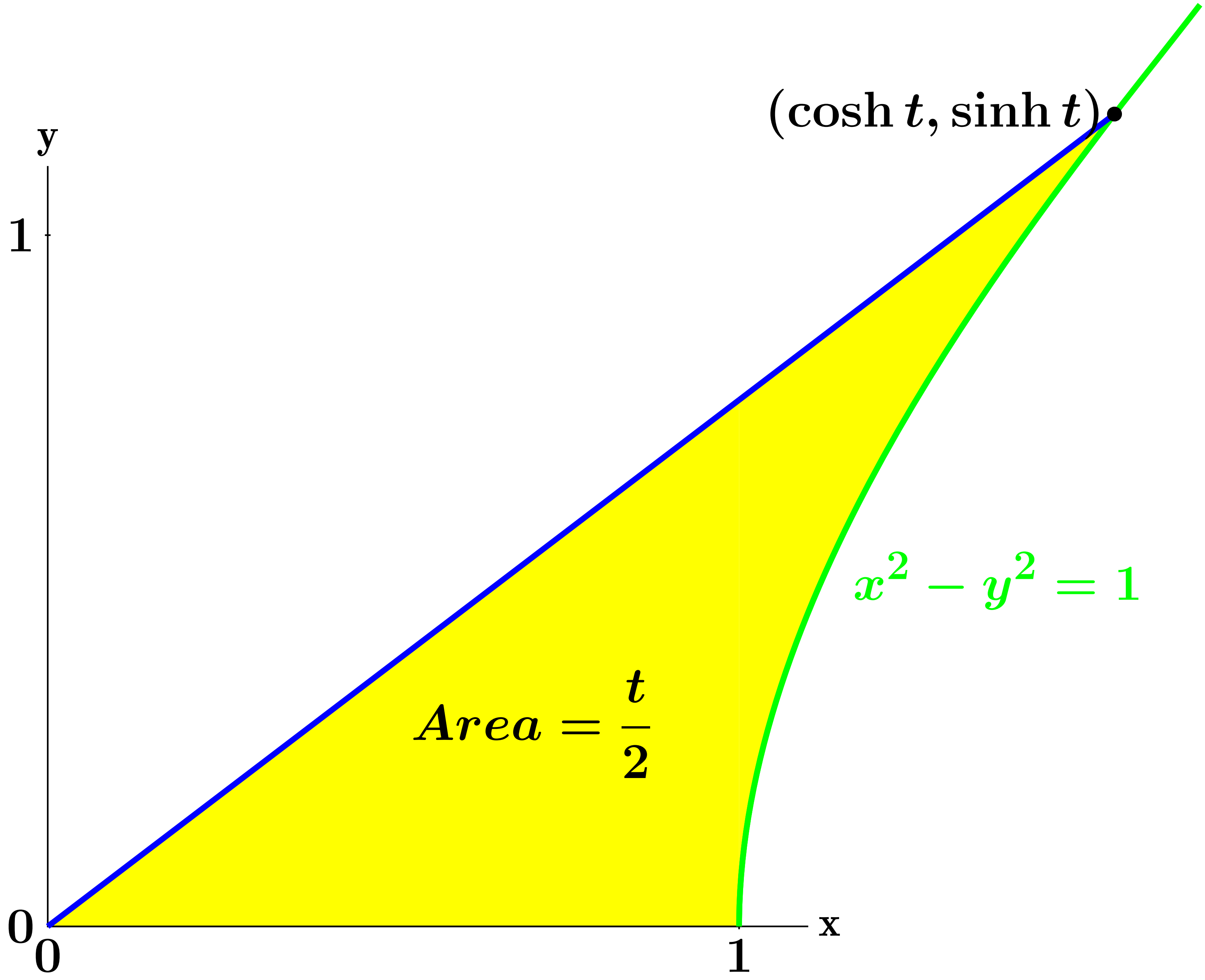
$$\boxed{\tfrac{d}{dx}\tanh x = \operatorname{sech}^2 x.}$$

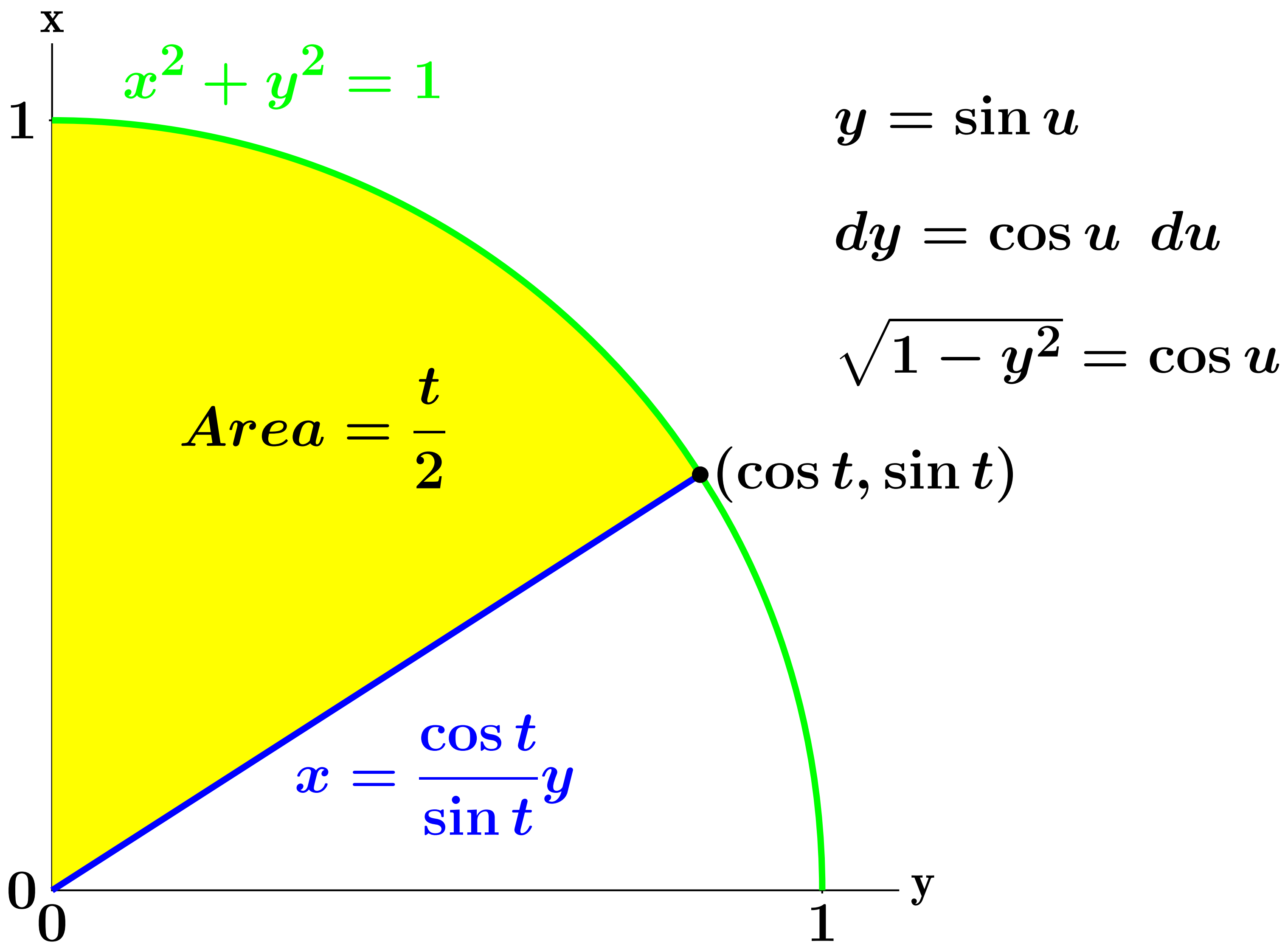
Proof.

$$\frac{d}{dx}\ \frac{\sinh x}{\cosh x} = \frac{(\sinh x)'\cosh x - (\cosh x)'\sinh x}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$


y
1
$(\cos t, \sin t)$
$x^2 + y^2 = 1$
$Area = \frac{t}{2}$
0
0
1
x


$(\cosh t, \sinh t)$
y
1
$x^2 - y^2 = 1$
$Area = \frac{t}{2}$
0
0
1
x


x
$x^2 + y^2 = 1$
1
$y = \sin u$
$dy = \cos u \;\; du$
$\sqrt{1 - y^2} = \cos u$
$Area = \frac{t}{2}$
$(\cos t, \sin t)$
$x = \frac{\cos t}{\sin t} y$
0
0
1
y

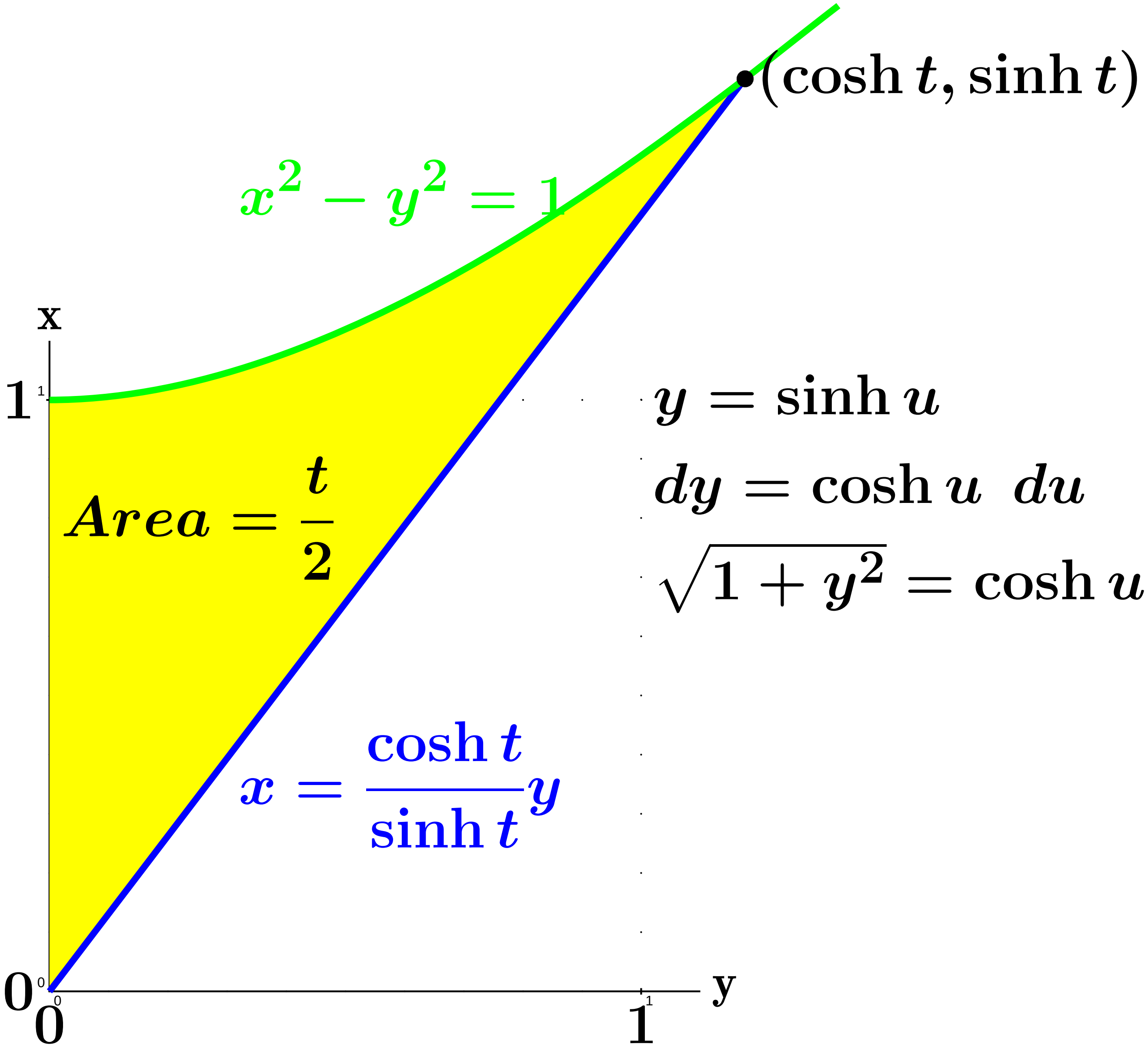
$$\int_{y=0}^{\sin t} \sqrt{1-y^2}\ dy \quad - \quad \int_{y=0}^{\sin t} \frac{\cos t}{\sin t} y\ dy$$

$$= \int_{u=0}^{t} \cos u \cos u\ du \quad - \quad \frac{\cos t}{\sin t} \int_{y=0}^{\sin t} y\ dy$$

$$= \frac{1}{2} \int_{u=0}^{t} 1 + \cos 2u\ du \quad - \quad \frac{\cos t}{\sin t} \left[\frac{y^2}{2} \right]_{y=0}^{\sin t}$$

$$= \frac{1}{2} \left(t \quad + \quad \frac{\sin 2t}{2} \right) \quad - \quad \frac{\cos t}{\sin t} \frac{\sin^2 t}{2}$$

$$= \frac{t}{2} \quad + \quad \frac{2 \sin t \cos t}{4} \quad - \quad \frac{\cos t \sin t}{2} \quad = \quad \frac{t}{2}$$



$$y = \sinh u$$

$$dy = \cosh u \;\; du$$

$$\sqrt{1 + y^2} = \cosh u$$

$$
\begin{aligned}
&\int_{y=0}^{\sinh t} \sqrt{1+y^2}\; dy \;-\; \int_{y=0}^{\sinh t} \frac{\cosh t}{\sinh t}\, y\; dy \\
&= \int_{u=0}^{t} \cosh^2 u\; du \;-\; \frac{\cosh t}{\sinh t} \int_{y=0}^{\sinh t} y\; dy \\
&= \int_{u=0}^{t} \frac{1+\cosh 2u}{2}\; du \;-\; \frac{\cosh t}{\sinh t} \left[\frac{y^2}{2}\right]_{y=0}^{\sinh t} \\
&= \frac{1}{2}\left(t + \frac{\sinh 2t}{2}\right) - \frac{\cosh t}{\sinh t}\,\frac{\sinh^2 t}{2} \\
&= \frac{t}{2} + \frac{2\sinh t\cosh t}{4} - \frac{\cosh t\sinh t}{2} = \frac{t}{2}
\end{aligned}
$$

$$\boxed{\sinh(s+t) = \sinh s \cosh t + \cosh s \sinh t}$$

Proof.

$$\frac{e^s - e^{-s}}{2}\ \frac{e^t + e^{-t}}{2} + \frac{e^s + e^{-s}}{2}\ \frac{e^t - e^{-t}}{2}$$

$$= \frac{e^s e^t - e^{-s} e^t + e^s e^{-t} - e^{-s} e^{-t}}{4}$$

$$+ \frac{e^s e^t + e^{-s} e^t - e^s e^{-t} - e^{-s} e^{-t}}{4}$$

$$= \frac{2\ e^s e^t - 2\ e^{-s} e^{-t}}{4}$$

$$= \frac{e^{(s+t)} - e^{-(s+t)}}{2}$$

$$\boxed{\sinh 2s = 2 \sinh s \cosh s}$$

$$\boxed{\cosh(s+t) = \cosh s \cosh t + \sinh s \sinh t}$$

Proof.
$$\frac{e^s + e^{-s}}{2}\;\frac{e^t + e^{-t}}{2} + \frac{e^s - e^{-s}}{2}\;\frac{e^t - e^{-t}}{2}$$

$$= \frac{e^s e^t + e^{-s} e^t + e^s e^{-t} + e^{-s} e^{-t}}{4}$$

$$+ \frac{e^s e^t - e^{-s} e^t - e^s e^{-t} + e^{-s} e^{-t}}{4}$$

$$= \frac{2\ \ e^s e^t + 2\ \ e^{-s} e^{-t}}{4} = \frac{e^{(s+t)} + \ \ e^{-(s+t)}}{2}$$

$$\begin{aligned} \cosh 2s &= \cosh^2 s + \sinh^2 s \\ &= 2\cosh^2 s - 1, \quad \text{using} \quad \sinh^2 s = \cosh^2 s - 1, \\ &= 1 + 2\sinh^2 s, \quad \text{using} \quad \cosh^2 s = 1 + \sinh^2 s. \end{aligned}$$

$$\begin{aligned} e^{x} &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ e^{-x} &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \end{aligned}$$

$$\frac{e^{x} + e^{-x}}{2} =$$

$$\begin{aligned} \cosh x &= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \\ \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \end{aligned}$$

$$\frac{e^{x} - e^{-x}}{2} =$$

$$\begin{aligned} \sinh x &= x + \frac{x^3}{6} + \frac{x^5}{120} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\ \sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$

Recall: $i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad$ **etc.**

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
e^{ix} &= 1 + ix - \frac{x^2}{2} - \frac{ix^3}{6} + \frac{x^4}{24} + \frac{ix^5}{120} + \dots = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} \\
e^{ix} &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots \\
&\quad + i\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \dots\right) = \cos x + i \sin x \\
\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots \\
\cos ix &= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \dots = \cosh x \\
\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} + \dots \\
\sin ix &= ix + \frac{ix^3}{6} + \frac{ix^5}{120} + \dots = i \sinh x
\end{aligned}
$$

We also have $\cosh ix = \cos x, \quad \sinh ix = i \sin x,$

$$\tan ix = i \tanh x, \quad \tanh ix = i \tan x.$$

$$e^{x} = \cosh x + \sinh x$$

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned}$$

$$\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \end{aligned}$$

$$\begin{aligned} e^{i\pi} &= \cos \pi + i \sin \pi = -1 \\ e^{i\pi} + 1 &= 0 \end{aligned}$$

If y equals arcsinh x, then x equals $\sinh y = \dfrac{e^y - e^{-y}}{2}$.

$$2x = e^y - e^{-y}$$

$$e^y - 2x - e^{-y} = 0$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x + \sqrt{x^2+1}$$

$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\boxed{\operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2+1}\right)}$$

$$(\operatorname{arcsinh} x)' = \frac{1 + \frac{1}{2}\frac{2x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} = \frac{\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} = \frac{1}{\sqrt{1+x^2}}$$

If y equals arccosh x, then x equals $\cosh y = \dfrac{e^y + e^{-y}}{2}$.

$$\begin{aligned} 2x &= e^y + e^{-y} \\ e^y - 2x + e^{-y} &= 0 \\ e^{2y} - 2xe^y + 1 &= 0 \\ e^y &= \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x + \sqrt{x^2 - 1} \\ y &= \ln\left(x + \sqrt{x^2 - 1}\right) \end{aligned}$$

$$\boxed{\text{arccosh } x = \ln\left(x + \sqrt{x^2 - 1}\right) \text{ for } x \ge 1}$$

$$(\text{arccosh } x)' = \frac{1 + \frac{1}{2}\frac{2x}{\sqrt{x^2 - 1}}}{x + \sqrt{x^2 - 1}} = \frac{\frac{\sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} + \frac{x}{\sqrt{x^2 - 1}}}{x + \sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

If y equals arctanh x, then x equals $\tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$.

$$\begin{aligned}
\left(e^y + e^{-y}\right)x &= e^y - e^{-y} \\
xe^y + xe^{-y} &= e^y - e^{-y} \\
0 &= (1-x)e^y - (1+x)e^{-y} \\
(1+x)e^{-y} &= (1-x)e^y \\
\frac{1+x}{1-x} &= e^{2y} \\
2y &= \ln\left(\frac{1+x}{1-x}\right) \\
y &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)
\end{aligned}$$

$$\boxed{\text{arctanh } x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \text{ for } -1 < x < 1}$$

$$
\begin{aligned}
(\operatorname{arctanh} x)' &= \left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right)' \\
&= \frac{1}{2}\big(\ln(1+x) - \ln(1-x)\big)' \\
&= \frac{1}{2}\left(\frac{1}{1+x} - \frac{-1}{1-x}\right) \\
&= \frac{1}{2}\,\frac{(1-x)+(1+x)}{(1+x)(1-x)} \\
&= \frac{1}{2}\,\frac{2}{(1+x)(1-x)} \\
(\operatorname{arctanh} x)' &= \frac{1}{1-x^2}
\end{aligned}
$$

$$\textbf{Integral Substitutions using } \begin{cases} a^2\cosh^2 t \;-\; a^2\sinh^2 t \;=\; a^2, \\[2ex] a^2\tanh^2 t \;+\; a^2\,\mathrm{sech}^2 t \;=\; a^2 \end{cases}$$

$$\begin{aligned}
\text{If } x &= a\sinh t : \\
\sqrt{x^2+a^2} &= \sqrt{a^2\sinh^2 t + a^2} \\
&= a\sqrt{\sinh^2 t + 1} \;=\; a\cosh t
\end{aligned}$$

$$\begin{aligned}
t \;=\; \operatorname{arcsinh}\frac{x}{a} &= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) \\
&= \ln\left(x + \sqrt{x^2+a^2}\right) - \ln a
\end{aligned}$$

$$\begin{aligned}
\textbf{If } x &= a\cosh t : \\
\sqrt{x^2 - a^2} &= \sqrt{a^2\cosh^2 t - a^2} \\
&= a\sqrt{\cosh^2 t - 1} \;=\; a\sinh t \\
t \;=\; \operatorname{arccosh}\frac{x}{a} &= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right) \\
&= \ln\left(x + \sqrt{x^2 - a^2}\right) - \ln a
\end{aligned}$$

$$\begin{aligned} \textbf{If } x &= a \tanh t : \\ \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \tanh^2 t} \\ &= a\sqrt{1 - \tanh^2 t} \;=\; a \operatorname{sech} t \\ t \;=\; \operatorname{arctanh}\frac{x}{a} &= \frac{1}{2}\ln\left(\frac{a+x}{a-x}\right) \end{aligned}$$

Every nonzero complex number $x + iy$ can be written as its absolute value $r = \sqrt{x^2 + y^2}$, which is real, times a number with absolute value equal to 1, which lies on the unit circle and which can be written as $\cos\theta + i\sin\theta$, where $\theta \pm 2k\pi i$ is a certain angle satisfying $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{x}{r}$. Call θ the argument of $z : \theta = \arg(z)$. Let ρ equal $\ln r$.

Since we have $z = x + iy = r(\cos\theta + i\sin\theta) = e^{\rho}e^{i\theta} = e^{\rho + i\theta}$, we can define the complex logarithm as a (multi- or single-valued) inverse of the complex exponential as follows:

$$\ln z = \ln x + iy = \rho + i\theta = \frac{1}{2}\ln(x^2 + y^2) + i\arg(z)$$

With this complex logarithm, the inverses of the (circular) trigonometric and hyperbolic functions can be listed:

$$\arcsin z = \frac{1}{i}\ln\left(z+\sqrt{1-z^2}\right), \quad \arccos z = \frac{1}{i}\ln\left(z+\sqrt{z^2-1}\right),$$

$$\arctan z = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right), \quad \mathrm{arc}\cot z = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right),$$

$$\operatorname{arcsec} z = \frac{1}{i}\ln\left(\frac{1+\sqrt{1-z^2}}{z}\right), \quad \mathrm{arc}\csc z = \frac{1}{i}\ln\left(\frac{1+\sqrt{z^2-1}}{z}\right),$$

$$\mathrm{arc}\sinh z = \ln\left(z+\sqrt{z^2+1}\right), \quad \mathrm{arccosh} z = \ln\left(z+\sqrt{z^2-1}\right),$$

$$\mathrm{arctanh} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \quad \mathrm{arccoth} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right),$$

$$\mathrm{arcsech} z = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right), \quad \mathrm{arccosech} z = \ln\left(\frac{1+\sqrt{z^2+1}}{z}\right).$$