$$\begin{cases} \textbf{If} & \lim\limits_{x\to c} f(x) \neq 0, \\ \textbf{and if} & \lim\limits_{x\to c} g(x) = 0, \end{cases} \quad \textbf{then } \lim_{x\to c} \frac{f(x)}{g(x)} = \pm\infty.$$

$$\begin{cases} \textbf{If} & \lim\limits_{x\to c} f(x) = 0, \\ \textbf{and if} & \lim\limits_{x\to c} g(x) = 0, \end{cases} \quad \textbf{then anything can happen:}$$

$$\begin{aligned} \lim_{x\to 0} \frac{x^2}{x} &= \lim_{x\to 0} x &&= 0, \\ \lim_{x\to 0} \frac{x}{x} &= \lim_{x\to 0} 1 &&= 1, \\ \lim_{x\to 0} \frac{x^2}{x^4} &= \lim_{x\to 0} \frac{1}{x^2} &&= \infty, \end{aligned}$$

There are two approaches:

Cancellation:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1 = 2$$

and L'Hospital's Rule:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{2x}{1} = 2.$$

L'Hospital's Rule $\left(\frac{0}{0}\text{ Version }\right)$:

$$\left\{\begin{array}{l}\text{If either } f \text{ or } g \text{ increases or decreases,}\\[1em] \text{if } f(x) \text{ approaches } 0,\\[1em] \text{if } g(x) \text{ approaches } 0, \text{ and}\\[1em] \text{if } \dfrac{f'(x)}{g'(x)} \text{ approaches } L, \qquad \text{as } x \text{ approaches } c(\pm),\end{array}\right.$$

$$\text{then } \frac{f(x)}{g(x)} \text{ also approaches } L.$$

An Old Example: $$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Another Example: $$\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} \frac{5\cos 5x}{1} = 5.$$

Still Another Example: $$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1.$$

A Warning: $\displaystyle\lim_{x\to 0} \frac{4x+3}{2x+5} = \frac{3}{5}$ **has nothing to do with**

$$\lim_{x\to 0} \frac{4}{2} = 2.$$

A Proof, for a limited hypothesis:

If f and g are differentiable (and thus continuous) at c,
if $f(c) = g(c) = 0$,
and if $g'(c) \neq 0$, then

$$\begin{aligned} f(x) &= f(c) + \big(f'(c) + o(1)\big)(x - c) \\ &= \phantom{f(c) + {}} \big(f'(c) + o(1)\big)(x - c) \end{aligned}$$

$$\begin{aligned} g(x) &= g(c) + \big(g'(c) + o(1)\big)(x - c) \\ &= \phantom{g(c) + {}} \big(g'(c) + o(1)\big)(x - c) \end{aligned}$$

$$\frac{f(x)}{g(x)} = \frac{\big(f'(c) + o(1)\big)(x - c)}{\big(g'(c) + o(1)\big)(x - c)} = \frac{f'(c) + o(1)}{g'(c) + o(1)} = \frac{f'(c)}{g'(c)} + o(1)$$

A Double Application of L'Hospital's Rule:

$$\lim_{x\to 0} \frac{1+x-e^x}{x(e^x-1)} \qquad \left(\to \frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{1-e^x}{e^x-1+xe^x} \qquad \left(\to \frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{-e^x}{e^x+e^x+xe^x} \qquad \left(\to \frac{-1}{2}\right)$$

Since $\dfrac{-e^x}{e^x+e^x+xe^x} \to -\dfrac{1}{2}$, **then so does** $\dfrac{1-e^x}{e^x-1+xe^x}$.

Since $\dfrac{1-e^x}{e^x-1+xe^x} \to -\dfrac{1}{2}$, **then so does** $\dfrac{1+x-e^x}{x(e^x-1)}$.

$$\lim_{x\to1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right)$$

$$=\lim_{x\to1}\frac{x-1-x\ln x}{(x-1)\ln x} \qquad \left(\to\frac{0}{0}\right)$$

$$=\lim_{x\to1}\frac{1-\ln x-1}{\ln x+\dfrac{x-1}{x}}$$

$$=\lim_{x\to1}\frac{-x\ln x}{x\ln x+x-1} \qquad \left(\to\frac{0}{0}\right)$$

$$=\lim_{x\to1}\frac{-\ln x-1}{\ln x+1+1} = -\frac{1}{2}.$$

Since this last limit exists and equals $-\frac{1}{2}$, so do the rest.

To evaluate a $\left(1^{\infty}\right)$ or a $\left(\infty^{0}\right)$ limit, work first on its logarithm:

$$\ln \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x} = \lim_{x\to\infty}\ln\left(1+\frac{a}{x}\right)^{x} = \lim_{x\to\infty} x\ln\left(1+\frac{a}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\ln\left(1+\dfrac{a}{x}\right)}{\dfrac{1}{x}} \qquad \left(\to\frac{0}{0}\right)$$

$$= \lim_{x\to\infty}\frac{\dfrac{1}{1+\dfrac{a}{x}}\cdot\left(-\dfrac{a}{x^{2}}\right)}{-\dfrac{1}{x^{2}}}$$

$$= \lim_{x\to\infty}\frac{a}{1+\dfrac{a}{x}} = a, \qquad \text{then unlog}$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x} = e^{a}$$

L'Hospital's Rule $\left(\dfrac{\infty}{\infty}\text{ Version }\right)$:

$$\left\{\begin{array}{l}\text{If either } f \text{ or } g \text{ increases or decreases,}\\ \text{if } f(x) \text{ approaches } \infty,\\ \text{if } g(x) \text{ approaches } \infty, \text{ and}\\ \text{if } \dfrac{f'(x)}{g'(x)} \text{ approaches } L,\end{array}\right. \qquad \text{as } x \text{ approaches } c(\pm),$$

$$\text{then } \frac{f(x)}{g(x)} \text{ also approaches } L.$$

$$\ln \lim_{x\to 0+} x^x = \lim_{x\to 0+} \ln x^x = \lim_{x\to 0+} x\ln x \quad (\to 0\cdot\infty)$$

$$= \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} \quad \left(\to \frac{\infty}{\infty}\right) = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0+} (-x) = 0$$

$$\lim_{x\to 0+} x^x = e^0 = 1$$

For $a > 0$, $\displaystyle \lim_{x\to\infty} \frac{\ln x}{x^a} = \lim_{x\to\infty} \frac{\frac{1}{x}}{ax^{a-1}} = \frac{1}{ax^a} = 0$,

yielding $\boxed{\ln x = o(x^a), \text{ for any } a > 0, \text{ as } x \to \infty.}$

$$\text{For } b > 0, \quad \lim_{u\to\infty} \frac{u^b}{e^u} = \lim_{x\to\infty} \frac{(\ln x)^b}{x}$$

$$= \lim_{x\to\infty} \left(\frac{\ln x}{x^{\frac{1}{b}}}\right)^b$$

$$= \left(\lim_{x\to\infty} \frac{\ln x}{x^{\frac{1}{b}}}\right)^b$$

$$= \left(0\right)^b$$

$$= 0,$$

yielding $\boxed{u^b = o(e^u), \text{ for any } b > 0, \text{ as } u \to \infty.}$

$$\lim_{x\to\infty}\left(\ln\left(1-\frac{1}{x}\right)\csc\frac{1}{x}\right) = \lim_{u\to 0+}\left(\ln(1-u)\csc u\right)$$

$$= \lim_{u\to 0+}\frac{\ln(1-u)}{\sin u}$$

$$= \lim_{u\to 0+}\frac{-\dfrac{1}{1-u}}{\cos u} = \frac{-1}{1} = -1$$

Special case: For positive integers n,

$$\lim_{n\to\infty}\left(\ln\left(1-\frac{1}{n}\right)\csc\frac{1}{n}\right) = -1$$

$$\begin{aligned}\lim_{x\to\infty} x\left(b^{\frac{1}{x}}-1\right) &= \lim_{x\to\infty}\frac{b^{\frac{1}{x}}-1}{\frac{1}{x}} = \lim_{u\to 0+}\frac{b^u-1}{u}\\ &= \lim_{u\to 0+}\frac{b^u\cdot\ln b}{1} = \ln b\end{aligned}$$

Special case: For positive integers n,

$$\lim_{n\to\infty} n\left(b^{\frac{1}{n}}-1\right) = \ln b$$

$$\ln \lim_{x \to \frac{\pi}{2}} |\sec x\ |^{\cos x} = \lim_{x \to \frac{\pi}{2}} \ln |\sec x\ |^{\cos x}$$

$$= \lim_{x \to \frac{\pi}{2}} \cos x \ln |\sec x\ | = -\lim_{x \to \frac{\pi}{2}} \cos x \ln |\cos x\ |$$

$$= -\lim_{u \to 0} u \ln |u\ | \quad (\to 0 \cdot \infty)$$

$$= -\lim_{u \to 0} \frac{\ln u}{\frac{1}{u}} \quad \left(\to \frac{\infty}{\infty}\right) \text{ or } = -\lim_{u \to 0} \frac{u}{\frac{1}{\ln u}} \quad \left(\to \frac{0}{0}\right)$$

$$= -\lim_{u \to 0} \frac{\frac{1}{u}}{-\frac{1}{u^2}} = \lim_{u \to 0} u = 0$$

$$\lim_{x \to \frac{\pi}{2}} |\sec x\ |^{\cos x} = e^0 = 1$$

Proof of L'Hospital's Rule $\left(\dfrac{0}{0}\text{ Version }\right)$:

Assume WOLG that the function g is increasing,

so that an inverse $h = g^{-1}$ exists.

Since $y = g(x)$ approaches 0 as x approaches c,

$h(y) = x$ also approaches c, as y approaches 0.

The inverse's derivative, $h'(y) = \dfrac{dx}{dy}$, equals $\dfrac{1}{\dfrac{dy}{dx}} = \dfrac{1}{g'(x)}$.

Since $\dfrac{f'(x)}{g'(x)}$ approaches L, as x approaches c, we have

$$\frac{d}{dy}f(h(y)) = f'(h(y))h'(y) = \frac{f'(h(y))}{g'(x)} = \frac{f'(x)}{g'(x)}$$

also approaching L, as y approaches 0.

For any positive ϵ, when $|x - c| <$ a certain δ, we have

$$L - \frac{\epsilon}{2} \;<\; \frac{d}{dy} f(h(y)) \;<\; L + \frac{\epsilon}{2}$$

Now take an x_1 properly between x and c,

and let y_1 equal $g(x_1)$, so that $x_1 = h(y_1)$.

(Remember that y equals $g(x)$ and that x equals $h(y)$.)

The Mean Value Theorem Corollary yields:

$$\begin{aligned} L - \frac{\epsilon}{2} &< \frac{f(h(y)) - f(h(y_1))}{y - y_1} < L + \frac{\epsilon}{2} \\ L - \frac{\epsilon}{2} &< \frac{f(x) - f(x_1)}{g(x) - g(x_1)} < L + \frac{\epsilon}{2} \end{aligned}$$

The Mean Value Theorem Corollary yields:

$$L - \frac{\epsilon}{2} \;<\; \frac{f(x) - f(x_1)}{g(x) - g(x_1)} \;<\; L + \frac{\epsilon}{2}$$

Let x_1 approach c, so that $f(x_1)$ and $g(x_1)$ can approach 0.

$$L - \epsilon \;<\; L - \frac{\epsilon}{2} \;\leq\; \frac{f(x) - 0}{g(x) - 0} \;\leq\; L + \frac{\epsilon}{2} \;<\; L + \epsilon$$

Thus $\dfrac{f(x)}{g(x)}$ approaches L.

Proof of L'Hospital's Rule $\left(\dfrac{\infty}{\infty}\text{ Version, for Left-hand Limits}\right)$:

As before, assume WOLG that g is increasing,
so that an inverse $h = g^{-1}$ exists.

Since $y = g(x)$ approaches ∞, as x approaches $c-$,
$h(y) = x$ also approaches $c-$, as y approaches ∞.

The inverse's derivative, $h'(y) = \dfrac{dx}{dy}$, equals $\dfrac{1}{\dfrac{dy}{dx}} = \dfrac{1}{g'(x)}$.

Since $\dfrac{f'(x)}{g'(x)}$ approaches L, as x approaches $c-$, we have

$$\frac{d}{dy}f(h(y)) = f'(h(y))h'(y) = \frac{f'(h(y))}{g'(x)} = \frac{f'(x)}{g'(x)}$$

also approaching L, as y approaches ∞.

For any positive ϵ, when $c - \delta < x < c$ for a certain δ, we have

$$L - \frac{\epsilon}{2} < \frac{d}{dy} f(h(y)) < L + \frac{\epsilon}{2}$$

Now take an x_1 such that $c - \delta < x_1 < x < c$,

let y_1 equal $g(x_1)$, so that $x_1 = h(y_1)$.

(Remember that y equals $g(x)$ and that x equals $h(y)$.)

The Mean Value Theorem Corollary yields:

$$L - \frac{\epsilon}{2} < \frac{f(h(y)) - f(h(y_1))}{y - y_1} < L + \frac{\epsilon}{2}$$

$$L - \frac{\epsilon}{2} < \frac{f(x) - f(x_1)}{g(x) - g(x_1)} < L + \frac{\epsilon}{2}$$

The Mean Value Theorem Corollary yields:

$$L-\frac{\epsilon}{2}<\frac{f(x)-f(x_1)}{g(x)-g(x_1)}<L+\frac{\epsilon}{2},$$

$$L-\frac{\epsilon}{2}<\frac{f(x)}{g(x)}\cdot\frac{1-\dfrac{f(x_1)}{f(x)}}{1-\dfrac{g(x_1)}{g(x)}}<L+\frac{\epsilon}{2},$$

$$\left(L-\frac{\epsilon}{2}\right)\cdot\frac{1-\frac{g(x_1)}{g(x)}}{1-\frac{f(x_1)}{f(x)}}<\frac{f(x)}{g(x)}<\left(L+\frac{\epsilon}{2}\right)\cdot\frac{1-\frac{g(x_1)}{g(x)}}{1-\frac{f(x_1)}{f(x)}},$$

As x approaches even closer to c,
and as $f(x)$ and $g(x)$ become larger and larger, we obtain

$$L-\epsilon<L-\frac{\epsilon}{2}\leq\frac{f(x)}{g(x)}\leq L+\frac{\epsilon}{2}<L+\epsilon,$$

and $\dfrac{f(x)}{g(x)}$ approaches L.

Notes:

In cases where $\dfrac{f'(x)}{g'(x)} \to \infty$, it is sufficient to replace the $(L-\epsilon)$s with M's, to replace the $(L-\frac{\epsilon}{2})$s with $(M+1)$'s and to forget about the $L+\epsilon$ inequalities on the right sides.

In cases where $\dfrac{f'(x)}{g'(x)} \to -\infty$, it is sufficient to replace the $(L+\epsilon)$s with $-M$'s, to replace the $(L+\frac{\epsilon}{2})$s with $(-M-1)$'s and to forget about the $L-\epsilon$ inequalities on the left sides.

It is important that either f or g be increasing or decreasing, here. Otherwise, L'Hospital's Theorem may not hold, as the following counterexample shows.

Counterexample:

Let $f(x)$ equal $\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)$,
which approaches ∞ as x approaches ∞.

Let $g(x)$ equal $\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)e^{\sin x}$,
which also approaches ∞ as x approaches ∞.

Now $\frac{f(x)}{g(x)}$ equals $e^{-\sin x}$, which has no limit, while

$$\begin{aligned}
\frac{f'(x)}{g'(x)} &= \frac{\left(\frac{1}{2}+\frac{\cos 2x}{2}\right)}{\left(\frac{1}{2}+\frac{\cos 2x}{2}\right)e^{\sin x}+\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)e^{\sin x}\cos x} \\
&= \frac{\cos^2 x}{\cos^2 x \;\; e^{\sin x}+\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)e^{\sin x}\cos x} \\
&= \frac{\cos x}{\cos x \;\; e^{\sin x}+\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)e^{\sin x}}
\end{aligned}$$

approaches 0, as x approaches ∞.

For a counterexample as u approaches c,

replace each x on the last two slides with $\dfrac{1}{u-c}$.

(It will not be a very pretty sight, typographically.)