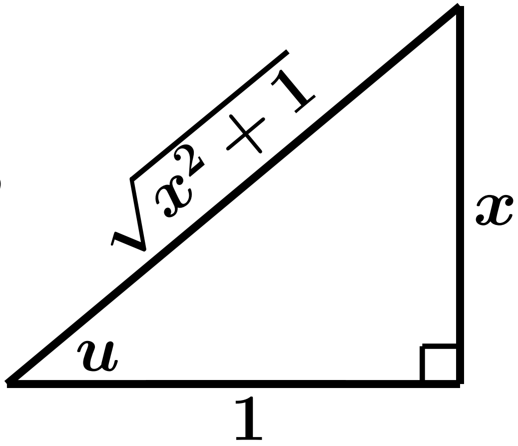
Triangle Setup for Handling $x^2 + 1$

$$x = \tan u, \qquad u = \arctan x,$$

$$dx = \sec^2 u \; du, \qquad \cos u = \frac{1}{\sqrt{x^2+1}},$$

$$x^2 + 1 = \sec^2 u, \qquad \sin u = \frac{x}{\sqrt{x^2+1}}.$$



$$\int \frac{1}{x^2+1}\; dx \;=\; \int \frac{1}{\sec^2 u}\; \sec^2 u \; du \;=\; \int du \;=\; u + C$$

$$= \arctan x + C$$

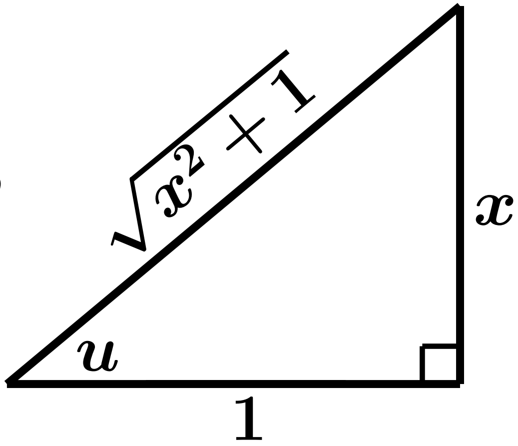
$$\int \frac{1}{(x^2+1)^2}\; dx \;=\; \int \frac{1}{\sec^4 u}\; \sec^2 u \; du \;=\; \int \frac{1}{\sec^2 u}\; du \ldots$$

Triangle Setup for Handling x^2+1

$x = \tan u,$

$dx = \sec^2 u \; du,$

$x^2 + 1 = \sec^2 u,$



$u = \arctan x,$

$\cos u = \dfrac{1}{\sqrt{x^2+1}},$

$\sin u = \dfrac{x}{\sqrt{x^2+1}}.$

$$\int \frac{1}{(x^2+1)^2}\, dx = \int \frac{1}{\sec^4 u} \sec^2 u \; du = \int \frac{1}{\sec^2 u}\, du$$

$$= \int \cos^2 u \; du = \int \frac{1+\cos 2u}{2}\, du = \frac{1}{2}\left(u + \frac{\sin 2u}{2}\right)$$

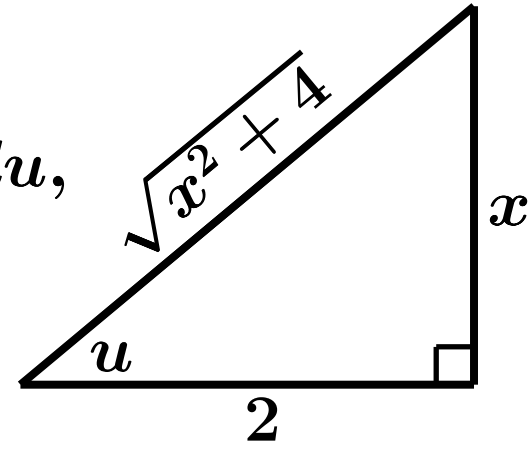
$$= \frac{1}{2}\left(u + \sin u \cos u\right) = \frac{1}{2}\left(\arctan x + \frac{1}{\sqrt{x^2+1}}\frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\arctan x + \frac{1}{2}\;\frac{x}{x^2+1} + C$$

Triangle Setup for Handling $x^2 + 4$

$$x = 2\tan u, \qquad u = \arctan\frac{x}{2},$$

$$dx = 2\sec^2 u \; du, \qquad \cos u = \frac{2}{\sqrt{x^2+4}},$$



$$x^2 + 4 = 4\sec^2 u, \qquad \sin u = \frac{x}{\sqrt{x^2+4}}.$$

$$\int \frac{1}{(x^2+4)^3}\; dx \;=\; \int \frac{1}{4^3\sec^6 u}\; 2\sec^2 u\; du \;=\; \int \frac{1}{32\sec^4 u}\; du$$

$$= \frac{1}{32}\int \cos^4 u\; du \;=\; \frac{1}{32}\left(\frac{3}{8}u + \frac{3}{8}\cos u \sin u + \frac{1}{4}\cos^3 u \sin u\right)$$

$$= \frac{1}{32}\left(\frac{3}{8}\arctan\frac{x}{2} + \frac{3}{8}\frac{2}{\sqrt{x^2+4}}\frac{x}{\sqrt{x^2+4}} + \frac{1}{4}\frac{8}{(x^2+4)^{3/2}}\frac{x}{\sqrt{x^2+4}}\right)$$

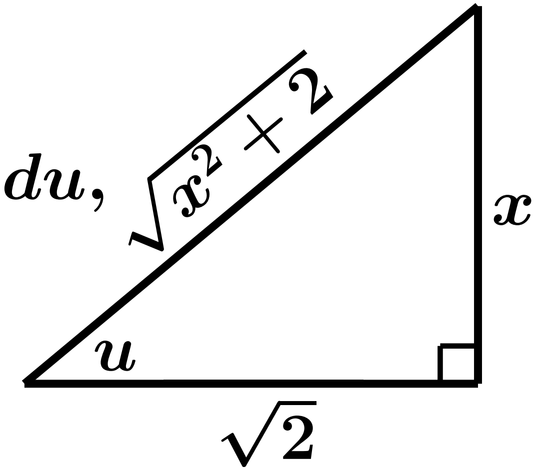
$$= \frac{3}{256}\arctan\frac{x}{2} + \frac{3x}{128(x^2+4)} + \frac{x}{16(x^2+4)^2} + C$$

Triangle Setup for Handling $x^2 + 2$

$$x = \sqrt{2}\tan u, \qquad u = \arctan\frac{x}{\sqrt{2}},$$

$$dx = \sqrt{2}\sec^2 u \; du, \qquad \cos u = \frac{\sqrt{2}}{\sqrt{x^2+2}},$$

$$x^2 + 2 = 2\sec^2 u, \qquad \sin u = \frac{x}{\sqrt{x^2+2}}.$$



$$\int \frac{1}{(x^2+2)^{3/2}}\; dx = \int \frac{1}{2^{3/2}\sec^3 u}\;\sqrt{2}\sec^2 u\; du$$

$$= \int \frac{1}{2\sec u}\; du = \frac{1}{2}\int \cos u\; du$$

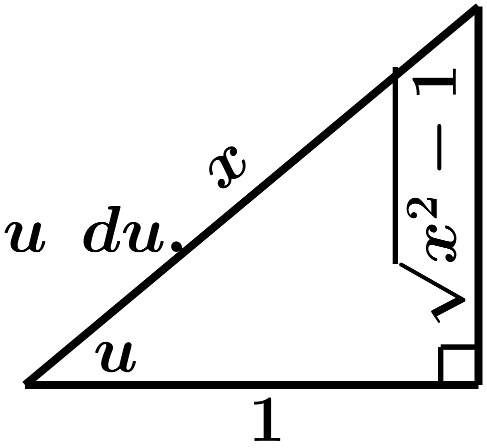
$$= \frac{1}{2}\sin u + C = \frac{1}{2}\frac{x}{\sqrt{x^2+2}} + C$$

Triangle Setup for Handling $x^2 - 1$

$$x = \sec u, \qquad u = \operatorname{arcsec} x,$$

$$dx = \sec u \tan u \; du, \qquad \sec u = x,$$

$$x^2 - 1 = \tan^2 u, \qquad \tan u = \sqrt{x^2 - 1}.$$



$$\int_{x=1}^{2} \sqrt{x^2 - 1} \; dx \;=\; \int_{\sec u = 1}^{\sec u = 2} \tan u \;\; \sec u \tan u \; du$$

$$= \int_{\cos u = 1}^{\cos u = \frac{1}{2}} \tan^2 u \sec u \; du \;=\; \int_{u=0}^{\frac{\pi}{3}} \left(\sec^2 u - 1 \right) \sec u \; du$$

$$= \int_{u=0}^{\frac{\pi}{3}} \sec^3 u - \sec u \; du$$

$$\int_{x=1}^{2} \sqrt{x^2-1}\ dx = \int_{\sec u=1}^{\sec u=2} \tan u \ \ \sec u \tan u \ du$$

$$= \int_{\cos u=1}^{\cos u=\frac{1}{2}} \tan^2 u \sec u \ du = \int_{u=0}^{\frac{\pi}{3}} \left(\sec^2 u - 1\right) \sec u \ du$$

$$= \int_{u=0}^{\frac{\pi}{3}} \sec^3 u - \sec u \ du$$

$$= \left[\frac{1}{2}\left(\sec u \tan u + \ln|\sec u + \tan u|\right) - \ln|\sec u + \tan u|\right]_{u=0}^{\frac{\pi}{3}}$$

$$= \left[\frac{1}{2}\left(\sec u \tan u - \ln|\sec u + \tan u|\right)\right]_{u=0}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[2\sqrt{3} - \ln\left|2+\sqrt{3}\right|\right] - \frac{1}{2}\left[0 - \ln|1+0|\right]$$

$$= \sqrt{3} - \frac{1}{2}\ln\left|2+\sqrt{3}\right|$$

or, if you are more comfortable with sines and cosines,

$$= \left[\frac{1}{2}\Big(\sec u \tan u - \ln|\sec u + \tan u|\Big)\right]_{u=0}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\frac{\sin u}{\cos^2 u} - \ln\left|\frac{1 + \sin u}{\cos u}\right|\right]_{u=0}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\frac{\frac{\sqrt{3}}{2}}{\left(\frac{1}{2}\right)^2} - \ln\left|\frac{1 + \frac{\sqrt{3}}{2}}{\frac{1}{2}}\right|\right] - \frac{1}{2}\left[0 - \ln\left|\frac{1+0}{1}\right|\right]$$

$$= \sqrt{3} - \frac{1}{2}\ln|2 + \sqrt{3}|$$

Hyperbolic Function Setup for Handling $x^2 - 1$

$$x = \cosh u, \qquad u = \operatorname{arccosh} x = \ln\left(x + \sqrt{x^2-1}\right),$$

$$dx = \sinh u \ du, \qquad \sqrt{x^2-1} = \sinh u.$$

$$\int_{x=1}^{2} \sqrt{x^2-1}\ dx = \int_{x=1}^{2} \sinh u \ \ \sinh u \ du$$

$$= \int_{x=1}^{2} \sinh^2 u \ du = \int_{x=1}^{2} \frac{\cosh 2u - 1}{2}\ du$$

$$= \frac{1}{2}\left[\frac{\sinh 2u}{2} - u\right]_{x=1}^{2} = \frac{1}{2}\left[\sinh u \cosh u - u\right]_{x=1}^{2}$$

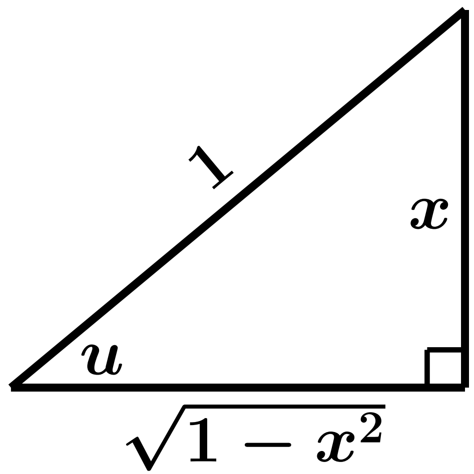
$$= \frac{1}{2}\left[x\sqrt{x^2-1} - \ln\left(x + \sqrt{x^2-1}\right)\right]_{x=1}^{2} = \sqrt{3} - \frac{1}{2}\ln\left|2+\sqrt{3}\right|$$

Triangle Setup for Handling $1 - x^2$

$$x = \sin u, \qquad u = \arcsin x,$$

$$dx = \cos u \; du, \qquad \cos u = \sqrt{1 - x^2},$$

$$1 - x^2 = \cos^2 u, \qquad \tan u = \frac{x}{\sqrt{1 - x^2}}.$$



$$\int \frac{x^2}{\left(1 - x^2\right)^{\frac{3}{2}}} \; dx \; = \; \int \frac{\sin^2 u}{\cos^3 u} \; \cos u \; du \; = \; \int \tan^2 u \; du$$

$$= \int \sec^2 u - 1 \; du \; = \; \tan u - u \; = \; \frac{x}{\sqrt{1 - x^2}} - \arcsin x + C$$

However, $\displaystyle\int \frac{x}{\left(1 - x^2\right)^{\frac{3}{2}}} \; dx$ **can use** $u = 1 - x^2$.

Before integrating anything involving a power of $9x^2-12x+29$ it will be necessary to complete the square:

Note:

The derivative of $9x^2-12x+29$ equals $18x-2=6(3x-2)$.

$$\begin{aligned}
9x^2-12x+29 &= (3x\ldots)^2\ldots \\
&= (3x-2)^2\ldots \\
&= 9x^2-12x+4+\ldots \\
&= 9x^2-12x+4+25 \\
&= (3x-2)^2+5^2
\end{aligned}$$

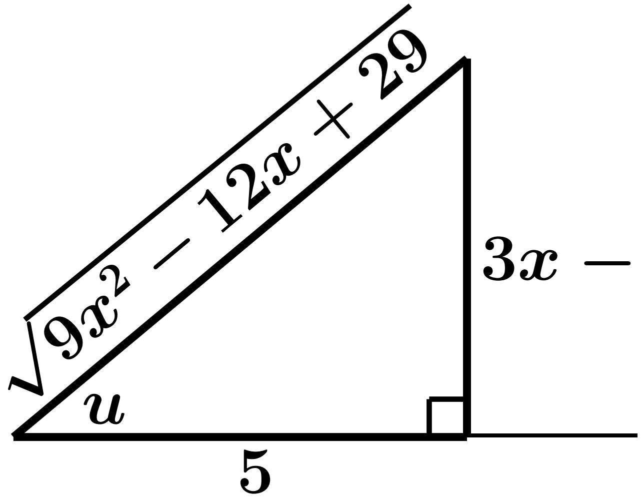
Triangle Setup for Handling $9x^2 - 12x + 29 = (3x-2)^2 + 5^2$:

$$\frac{3x-2}{5} = \tan u, \qquad u = \arctan\frac{3x-2}{5},$$

$$3x - 2 = 5\tan u,$$

$$dx = \frac{5}{3}\sec^2 u \; du,$$

$$9x^2 - 12x + 29 = 25\sec^2 u,$$



$$\cos u = \frac{5}{\sqrt{9x^2-12x+29}},$$

$$\sin u = \frac{3x-2}{\sqrt{9x^2-12x+29}}.$$

$$\begin{aligned}
\int \frac{1}{9x^2 - 12x + 29}\ dx &= \int \frac{1}{25\sec^2 u}\ \frac{5}{3}\sec^2 u\ du \\
&= \frac{1}{15}\int du \\
&= \frac{1}{15}u + C \\
&= \frac{1}{15}\arctan\frac{3x-2}{5} + C
\end{aligned}$$

$$\int \frac{1}{\left(9x^2 - 12x + 29\right)^2}\ dx \ = \ \int \frac{1}{\left(25\sec^2 u\right)^2}\ \frac{5}{3}\sec^2 u\ \ du$$

$$= \ \int \frac{1}{625\sec^4 u}\ \frac{5}{3}\sec^2 u\ \ du \ = \ \frac{5}{625\cdot 3}\int \cos^2 u\ \ du$$

$$= \frac{1}{375}\int \frac{1+\cos 2u}{2}\ du \ = \ \frac{1}{750}\int 1+\cos 2u\ \ du$$

$$= \frac{1}{750}\left(u + \frac{\sin 2u}{2}\right) \ = \ \frac{1}{750}\left(u + \sin u \cos u\right) + C$$

$$= \frac{1}{750}\left(\tan^{-1}\frac{3x-2}{5} + \frac{5}{\sqrt{9x^2-12x+29}}\frac{3x-2}{\sqrt{9x^2-12x+29}}\right) + C$$

$$= \frac{1}{750}\arctan\frac{3x-2}{5} \ + \ \frac{1}{150}\ \frac{3x-2}{9x^2-12x+29} + C$$

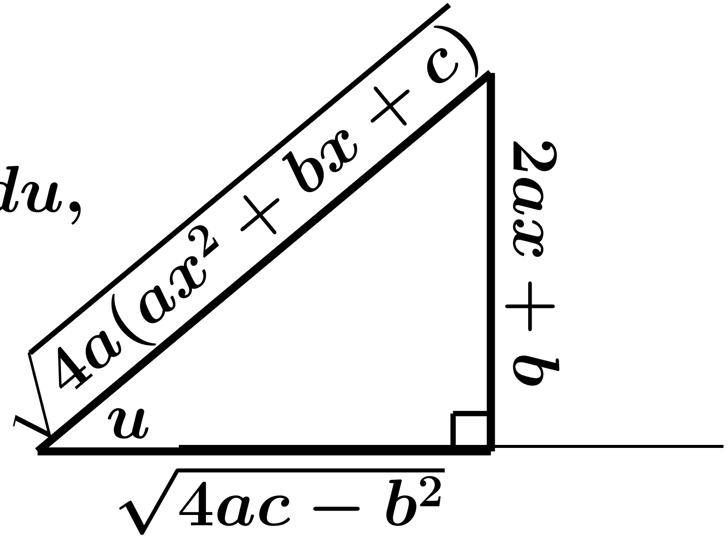
Triangle Setup for $ax^2+bx+c=\dfrac{(2ax+b)^2+(4ac-b^2)}{4a}$:

$$\frac{2ax+b}{\sqrt{4ac-b^2}}=\tan u,\qquad u=\arctan\frac{2ax+b}{\sqrt{4ac-b^2}},$$

$$2ax+b=\sqrt{4ac-b^2}\tan u,$$

$$dx=\frac{\sqrt{4ac-b^2}}{2a}\sec^2 u\ \ du,$$

$$ax^2+bx+c=\frac{4ac-b^2}{4a}\sec^2 u,$$



$$\cos u=\frac{\sqrt{4ac-b^2}}{\sqrt{4a(ax^2+bx+c)}},$$

$$\sin u=\frac{2ax+b}{\sqrt{4a(ax^2+bx+c)}}.$$

$$\int \frac{1}{ax^2 + bx + c} \, dx$$

$$= \int \frac{1}{\frac{4ac - b^2}{4a} \sec^2 u} \, \frac{\sqrt{4ac - b^2}}{2a} \sec^2 u \, du$$

$$= \frac{2}{\sqrt{4ac - b^2}} \int du$$

$$= \frac{2}{\sqrt{4ac - b^2}} \, u + C$$

$$= \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}} \; + \; C$$

$$\int \frac{1}{\left(ax^2+bx+c\right)^2}\ dx \ = \ \int \frac{\frac{\sqrt{4ac-b^2}}{2a}}{\left(\frac{4ac-b^2}{4a}\sec^2 u\right)^2}\sec^2 u\ \ du$$

$$= \ \int \frac{8a}{(4ac-b^2)^{\frac{3}{2}}\sec^4 u}\ \sec^2 u\ \ du \ = \ \frac{8a}{(4ac-b^2)^{\frac{3}{2}}}\int \cos^2 u\ \ du$$

$$= \frac{8a}{(4ac-b^2)^{\frac{3}{2}}}\int \frac{1+\cos 2u}{2} du = \frac{4a}{(4ac-b^2)^{\frac{3}{2}}}\int 1+\cos 2u du$$

$$= \frac{4a}{(4ac-b^2)^{\frac{3}{2}}}\left(u+\frac{\sin 2u}{2}\right) = \frac{4a}{(4ac-b^2)^{\frac{3}{2}}}\left(u+\sin u\cos u\right)+C$$

$$= \frac{4a}{(4ac-b^2)^{\frac{3}{2}}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}}$$

$$+ \ \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} \ + \ C$$