

We shall start with:

$$\int_{t=a}^x f'(t) dt = [f(t)]_{t=a}^x = f(x) - f(a),$$

then rearrange:

$$f(x) = f(a) + \int_{t=a}^x f'(t) dt$$

$$f(x) = f(a) - \int_{t=a}^x f'(t)(x-t)' dt,$$

then integrate by parts:

$$f(x) = f(a) - [f'(t)(x-t)]_{t=a}^x + \int_{t=a}^x f''(t)(x-t) dt,$$

$$f(x) = f(a) - f'(x)(x-x) + f'(a)(x-a) + \int_{t=a}^x f''(t)(x-t) dt,$$

$$f(x) = f(a) + f'(a)(x-a) + \int_{t=a}^x f''(t)(x-t) dt,$$

$$f(x) = f(a) + f'(a)(x - a) + \int_{t=a}^x f''(t)(x - t) dt,$$

$$f(x) = f(a) + f'(a)(x - a) - \int_{t=a}^x f''(t) \left(\frac{(x - t)^2}{2} \right)' dt,$$

then integrate by parts again:

$$f(x) = f(a) + f'(a)(x - a) - \left[f''(t) \frac{(x - t)^2}{2} \right]_{t=a}^x + \int_{t=a}^x f^{(3)}(t) \frac{(x - t)^2}{2} dt,$$

$$f(x) = f(a) + f'(a)(x - a) + f''(a) \frac{(x - a)^2}{2} + \int_{t=a}^x f^{(3)}(t) \frac{(x - t)^2}{2} dt,$$

$$f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2} + \int_{t=a}^x f^{(3)}(t)\frac{(x - t)^2}{2} dt,$$

$$f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2!} - \int_{t=a}^x f^{(3)}(t)\left(\frac{(x - t)^3}{3!}\right)' dt,$$

then integrate by parts again:

$$f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2!} + f^{(3)}(a)\frac{(x - a)^3}{3!} + \int_{t=a}^x f^{(4)}(t)\frac{(x - t)^3}{3!} dt,$$

$$f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2!} + f^{(3)}(a)\frac{(x - a)^3}{3!} + \int_{t=a}^x f^{(4)}(t)\frac{(x - t)^3}{3!} dt,$$

then integrate by parts again:

$$f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2!} + f^{(3)}(a)\frac{(x - a)^3}{3!} + f^{(4)}(a)\frac{(x - a)^4}{4!} + \int_{t=a}^x f^{(5)}(t)\frac{(x - t)^4}{4!} dt,$$

and again:

$$f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2!} + f^{(3)}(a)\frac{(x - a)^3}{3!} + f^{(4)}(a)\frac{(x - a)^4}{4!} + f^{(5)}(a)\frac{(x - a)^5}{5!} + \int_{t=a}^x f^{(6)}(t)\frac{(x - t)^5}{5!} dt.$$

If f can be differentiated $n + 1$ times we have

$$f(x) = \sum_{k=0}^n f^{(k)}(a) \frac{(x-a)^k}{k!} + \int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt.$$

(Taylor Polynomial) (Integral Remainder)

Since $f^{(n+1)}$ satisfies the Intermediate Value Property,
the Integral Mean Value Theorem yields

$$\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt = f^{(n+1)}(c) \int_{t=a}^x \frac{(x-t)^n}{n!} dt,$$

where c is some unknown number between a and x .

If f can be differentiated $n + 1$ times we have

$$f(x) = \sum_{k=0}^n f^{(k)}(a) \frac{(x-a)^k}{k!} + f^{(n+1)}(c) \frac{(x-a)^{n+1}}{(n+1)!}.$$

(Taylor Polynomial) (Lagrange Remainder)

We shall bound the Integral Remainder: $\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt.$

If, when t is between a and x ,
 we have $f^{(n+1)}(t)$ lying between minimum m and maximum M ,

then we also have

$$f^{(n+1)}(t) \frac{(x-t)^n}{n!} \quad \text{lying between } m \frac{(x-t)^n}{n!}$$

$$\text{and } M \frac{(x-t)^n}{n!},$$

$$\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \quad \text{lying between } \int_{t=a}^x m \frac{(x-t)^n}{n!} dt$$

$$\text{and } \int_{t=a}^x M \frac{(x-t)^n}{n!} dt,$$

$$\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \text{ lying between } m \int_{t=a}^x \frac{(x-t)^n}{n!} dt$$

and $M \int_{t=a}^x \frac{(x-t)^n}{n!} dt,$

$$\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \text{ lying between } m \left[-\frac{(x-t)^{n+1}}{(n+1)!} \right]_{t=a}^x$$

and $M \left[-\frac{(x-t)^{n+1}}{(n+1)!} \right]_{t=a}^x,$

$$\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \text{ lying between } m \frac{(x-a)^{n+1}}{(n+1)!}$$

and $M \frac{(x-a)^{n+1}}{(n+1)!},$

$$\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

lying between $m \frac{(x-a)^{n+1}}{(n+1)!}$
and $M \frac{(x-a)^{n+1}}{(n+1)!}$,

$$\frac{\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt}{\frac{(x-a)^{n+1}}{(n+1)!}}$$

lying between m and M ,

just like the function $f^{(n+1)}(x)$ does.

Now if $f^{(n+1)}(x)$ has maximum M and minimum m ,

then we can have
$$\frac{\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt}{\frac{(x-a)^{n+1}}{(n+1)!}}$$

equal to $f^{(n+1)}(c)$ for some c between a and x .

This is why Lagrange wrote

$$\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt = f^{(n+1)}(c) \frac{(x-a)^{n+1}}{(n+1)!}$$

(Easier to write)

Using this, if we know that $f^{(n+1)}(t)$ is between m and M , then we know that

$$\int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \text{ lies between } m \frac{(x-a)^{n+1}}{(n+1)!}$$

and $M \frac{(x-a)^{n+1}}{(n+1)!}$.

(Easier to use)

If f can be differentiated $n + 1$ times we have

$$f(x) = \sum_{k=0}^n f^{(k)}(a) \frac{(x-a)^k}{k!} + \int_{t=a}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt.$$

(Taylor Polynomial) (Integral Remainder)

now means

If f can be differentiated $n + 1$ times, with $m \leq f^{(n+1)}(t) \leq M$,
then $f(x)$ lies between

$$\sum_{k=0}^n f^{(k)}(a) \frac{(x-a)^k}{k!} + m \frac{(x-a)^{n+1}}{(n+1)!}$$

and

$$\sum_{k=0}^n f^{(k)}(a) \frac{(x-a)^k}{k!} + M \frac{(x-a)^{n+1}}{(n+1)!}$$

Example: Let $f(x)$ equal $x^4 - x^3 + x^2 - x + 1$. and let $a = 2$.

$$f(x) = x^4 - x^3 + x^2 - x + 1. \quad f(2) = 16 - 8 + 4 - 2 + 1 = 11.$$

$$f'(x) = 4x^3 - 3x^2 + 2x - 1. \quad f'(2) = 32 - 12 + 4 - 1 = 23.$$

$$f''(x) = 12x^2 - 6x + 2. \quad f''(2) = 48 - 12 + 2 = 38.$$

$$f^{(3)}(x) = 24x - 6. \quad f^{(3)}(2) = 48 - 6 = 42.$$

$$f^{(4)}(x) \equiv 24. \quad 24 \leq f^{(4)}(c) \leq 24.$$

The Third Degree Taylor Polynomial is

$$\begin{aligned} & 11 \frac{(x-2)^0}{0!} + 23 \frac{(x-2)^1}{1!} + 38 \frac{(x-2)^2}{2!} + 42 \frac{(x-2)^3}{3!} \\ & = 11 + 23(x-2) + 19(x-2)^2 + 7(x-2)^3. \end{aligned}$$

$f(x)$ lies between

$$\begin{aligned} & 11 + 23(x-2) + 19(x-2)^2 + 7(x-2)^3 + 24 \frac{(x-2)^4}{4!} \\ & \text{and } 11 + 23(x-2) + 19(x-2)^2 + 7(x-2)^3 + 24 \frac{(x-2)^4}{4!}. \end{aligned}$$

Example: Let $f(x)$ equal $x^4 - x^3 + x^2 - x + 1$. and let $a = 2$.

$$f(x) = x^4 - x^3 + x^2 - x + 1. \quad f(2) = 16 - 8 + 4 - 2 + 1 = 11.$$

$$f'(x) = 4x^3 - 3x^2 + 2x - 1. \quad f'(2) = 32 - 12 + 4 - 1 = 23.$$

$$f''(x) = 12x^2 - 6x + 2. \quad f''(2) = 48 - 12 + 2 = 38.$$

$$f^{(3)}(x) = 24x - 6. \quad f^{(3)}(2) = 48 - 6 = 42.$$

$$f^{(4)}(x) = 24. \quad f^{(4)}(2) = 24.$$

$$f^{(5)}(x) \equiv 0. \quad 0 \leq f^{(5)}(c) \leq 0$$

The Fourth Degree Taylor Polynomial is

$$11 + 23(x - 2) + 19(x - 2)^2 + 7(x - 2)^3 + (x - 2)^4.$$

$f(x)$ lies between

$$11 + 23(x - 2) + 19(x - 2)^2 + 7(x - 2)^3 + (x - 2)^4 + 0 \frac{(x - 2)^5}{5!},$$

$$11 + 23(x - 2) + 19(x - 2)^2 + 7(x - 2)^3 + (x - 2)^4 + 0 \frac{(x - 2)^5}{5!}.$$

Example: Let $f(x)$ equal $\sin x$ and let a equal 0.

$$f(x) = \sin x. \quad f(0) = \sin 0 = 0.$$

$$f'(x) = \cos x. \quad f'(0) = \cos 0 = 1.$$

$$f''(x) = -\sin x. \quad f''(0) = -\sin 0 = 0.$$

$$f^{(3)}(x) = -\cos x. \quad f^{(3)}(0) = -\cos 0 = -1.$$

$$f^{(4)}(x) = \sin x. \quad f^{(4)}(c) = \sin c.$$

Third Degree Taylor Polynomial:

$$f(x) = 0 \frac{x^0}{0!} + 1 \frac{x^1}{1!} + 0 \frac{x^2}{2!} - 1 \frac{x^3}{3!} = x - \frac{x^3}{3!}$$

If $0 < x < \frac{\pi}{2}$, then we have

$$0 < c < x < \frac{\pi}{2},$$
$$\sin 0 < \sin c < \sin \frac{\pi}{2},$$
$$0 < f^{(4)}(c) < 1,$$

If $0 < x < \frac{\pi}{2}$, then we have $0 < f^{(4)}(c) < 1$,

The Lagrange Remainder satisfies: $0 < f^{(4)}(c) \frac{x^4}{4!} < 1 \cdot \frac{x^4}{4!}$,

so that $\sin x = f(x) = x - \frac{x^3}{3!} + f^{(4)}(c) \frac{x^4}{4!}$

lies between $x - \frac{x^3}{3!} + 0$

and $x - \frac{x^3}{3!} + \frac{x^4}{4!}$.

If x were to equal $\frac{3}{10}$,

this would place $\sin \frac{3}{10}$ between 0.2955 and 0.2957.

Example: Let $f(x)$ equal $\sin x$ and let a equal 0.

$$f(x) = \sin x. \quad f(0) = \sin 0 = 0.$$

$$f'(x) = \cos x. \quad f'(0) = \cos 0 = 1.$$

$$f''(x) = -\sin x. \quad f''(0) = -\sin 0 = 0.$$

$$f^{(3)}(x) = -\cos x. \quad f^{(3)}(0) = -\cos 0 = -1.$$

$$f^{(4)}(x) = \sin x. \quad f^{(4)}(0) = \sin 0 = 0.$$

$$f^{(5)}(x) = \cos x. \quad f^{(5)}(c) = \cos c.$$

Fourth Degree Taylor Polynomial: (same as Third Degree)

$$f(x) = 0 \frac{x^0}{0!} + 1 \frac{x^1}{1!} + 0 \frac{x^2}{2!} - 1 \frac{x^3}{3!} + 0 \frac{x^4}{4!} = x - \frac{x^3}{3!}$$

If $0 < x < \frac{\pi}{2}$, then we have

$$0 < c < x < \frac{\pi}{2},$$
$$\cos 0 > \cos c > \cos \frac{\pi}{2},$$
$$1 > f^{(5)}(c) > 0,$$

If $0 < x < \frac{\pi}{2}$, then we have $0 < f^{(5)}(c) < 1$,

This Lagrange Remainder satisfies: $0 < f^{(5)}(c) \frac{x^5}{5!} < 1 \cdot \frac{x^5}{5!}$,

$$\text{so that } \sin x = f(x) = x - \frac{x^3}{3!} + f^{(5)}(c) \frac{x^5}{5!}$$

$$\text{lies between } x - \frac{x^3}{3!} + 0$$

$$\text{and } x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

If x were to equal $\frac{3}{10}$,

this would place $\sin \frac{3}{10}$ between 0.29550 and 0.29553.

Example: Let $f(x)$ equal $\sin x$ and let a equal 0.

$$\begin{array}{ll}
 f^{(4k)}(x) = \sin x. & f^{(4k)}(0) = \sin 0 = 0. \\
 f^{(4k+1)}(x) = \cos x. & f^{(4k+1)}(0) = \cos 0 = 1. \\
 f^{(4k+2)}(x) = -\sin x. & f^{(4k+2)}(0) = -\sin 0 = 0. \\
 f^{(4k+3)}(x) = -\cos x. & f^{(4k+3)}(0) = -\cos 0 = -1.
 \end{array}$$

From the above we have

$$\begin{array}{ll}
 f^{(2L)}(x) = (-1)^L \sin x. & f^{(2L)}(0) = (-1)^L \sin 0 = 0. \\
 f^{(2L+1)}(x) = (-1)^L \cos x. & f^{(2L+1)}(0) = (-1)^L \cos 0 = (-1)^L.
 \end{array}$$

The Taylor Polynomial of degree $2L$:

$$\begin{array}{l}
 0 \frac{x^0}{0!} + 1 \frac{x^1}{1!} + 0 \frac{x^2}{2!} - 1 \frac{x^3}{3!} + \cdots + (-1)^{L-1} \frac{x^{2L-1}}{(2L-1)!} + 0 \cdot \frac{x^{2L}}{(2L)!} \\
 \text{equals } x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots + (-1)^{L-1} \frac{x^{2L-1}}{(2L-1)!}
 \end{array}$$

If, when x is between 0 and $\frac{\pi}{2}$,

we have $\cos x$ lying between 0 and 1,

$f^{(2L+1)}(c) = (-1)^L \cos c$ lying between 0 and $(-1)^L$,

$f^{(2L+1)}(c) \frac{x^{(2L+1)}}{(2L+1)!}$ lying between 0 and $(-1)^L \frac{x^{(2L+1)}}{(2L+1)!}$,

so that $\sin x = f(x)$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{L-1} \frac{x^{2L-1}}{(2L-1)!} + f^{(2L+1)}(c) \frac{x^{(2L+1)}}{(2L+1)!}$$

lies between

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{L-1} \frac{x^{2L-1}}{(2L-1)!}$$

and $x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{L-1} \frac{x^{2L-1}}{(2L-1)!} + (-1)^L \frac{x^{(2L+1)}}{(2L+1)!}$,

two consecutive Taylor Polynomials.

For $0 < x < \frac{\pi}{2}$, the value of $\sin x$ lies between
the values of any two consecutive Taylor Polynomials.
of odd degree.

This also apparent from the fact that

the entire infinite series $\sum_{L=0}^{\infty} (-1)^L \frac{x^{(2L+1)}}{(2L+1)!}$ for the sine
is an alternating series for $0 < x < \frac{\pi}{2}$.

For any other value of x ,

the series terms start decreasing in size after L passes $\frac{|x|}{2}$,
and they start to form an alternating tail of this series.

Then $\sin x$ will be straddled by the partial sums of this series.

Example: Let $f(x)$ equal $\cos x$ and let a equal 0.

$$\begin{array}{ll}
 f^{(4k)}(x) = \cos x. & f^{(4k)}(0) = \cos 0 = 1. \\
 f^{(4k+1)}(x) = -\sin x. & f^{(4k+1)}(0) = -\sin 0 = 0. \\
 f^{(4k+2)}(x) = -\cos x. & f^{(4k+2)}(0) = -\cos 0 = -1. \\
 f^{(4k+3)}(x) = \sin x. & f^{(4k+3)}(0) = \sin 0 = 0.
 \end{array}$$

From the above we have

$$\begin{array}{ll}
 f^{(2L)}(x) = (-1)^L \cos x. & f^{(2L)}(0) = (-1)^L \cos 0 = (-1)^L. \\
 f^{(2L+1)}(x) = (-1)^{L+1} \sin x. & f^{(2L+1)}(0) = (-1)^{L+1} \sin 0 = 0.
 \end{array}$$

The Taylor Polynomial of degree $2L - 1$ equals

$$\begin{array}{l}
 1 \frac{x^0}{0!} + 0 \frac{x^1}{1!} - 1 \frac{x^2}{2!} + 0 \frac{x^3}{3!} + \cdots + (-1)^{L-1} \frac{x^{2L-2}}{(2L-2)!} + 0 \frac{x^{2L-1}}{(2L-1)!} \\
 \text{equals } 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots + (-1)^{L-1} \frac{x^{2L-2}}{(2L-2)!}
 \end{array}$$

If, when x is between 0 and $\frac{\pi}{2}$,

we have $\cos x$ lying between 0 and 1,

$f^{(2L)}(c) = (-1)^L \cos c$ lying between 0 and $(-1)^L$,

$f^{(2L)}(c) \frac{x^{2L}}{(2L)!}$ lying between 0 and $(-1)^L \frac{x^{2L}}{(2L)!}$,

so that $\cos x = f(x)$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^{L-1} \frac{x^{2L-2}}{(2L-2)!} + f^{(2L)}(c) \frac{x^{2L}}{(2L)!}$$

lies between

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^{L-1} \frac{x^{2L-2}}{(2L-2)!}$$

$$\text{and } 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^{L-1} \frac{x^{2L-2}}{(2L-2)!} + (-1)^L \frac{x^{2L}}{(2L)!},$$

two consecutive Taylor Polynomials.

For $0 < x < \frac{\pi}{2}$, the value of $\cos x$ lies between
the values of any two consecutive Taylor Polynomials
of even degree.

This also apparent from the fact that

the infinite series $\sum_{L=0}^{\infty} (-1)^L \frac{x^{2L}}{(2L)!}$ for the cosine

is an alternating series for $0 < x < \frac{\pi}{2}$.

For any other value of x ,

the series terms start decreasing in size after L passes $\frac{|x|}{2}$,
and they start to form an alternating tail of this series.

Then $\cos x$ will be straddled by the partial sums of this series.

Example: Let $f(x)$ equal e^x and let a equal 0.

$$f^{(k)}(x) = e^x. \quad f^{(k)}(0) = e^0 = 1.$$

Taylor Polynomial of Degree n equals

$$\begin{aligned} & 1 \frac{x^0}{0!} + 1 \frac{x^1}{1!} + 1 \frac{x^2}{2!} + 1 \frac{x^3}{3!} + \dots + 1 \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} = \sum_{k=0}^n \frac{x^k}{k!} \end{aligned}$$

If $0 < x$, then $0 < c < x$ and $1 = e^0 < f^{(n+1)}(c) = e^c < e^x$.

The Lagrange Remainder is bounded by

$$1 \cdot \frac{x^{n+1}}{(n+1)!} < f^{(n+1)}(c) \frac{x^{n+1}}{(n+1)!} < e^x \frac{x^{n+1}}{(n+1)!}$$

If x is positive we now have

$$\frac{x^{n+1}}{(n+1)!} < f^{(n+1)}(c) \frac{x^{n+1}}{(n+1)!} < e^x \frac{x^{n+1}}{(n+1)!}.$$

With $e^x = \sum_{k=0}^n \frac{x^k}{k!} + f^{(n+1)}(c) \frac{x^{n+1}}{(n+1)!}$ this becomes

$$\sum_{k=0}^n \frac{x^k}{k!} + \frac{x^{n+1}}{(n+1)!} < e^x < \sum_{k=0}^n \frac{x^k}{k!} + e^x \frac{x^{n+1}}{(n+1)!}.$$

The right-side inequality is equivalent to $e^x - e^x \frac{x^{n+1}}{(n+1)!} < \sum_{k=0}^n \frac{x^k}{k!}$,

$$e^x \left(1 - \frac{x^{n+1}}{(n+1)!} \right) < \sum_{k=0}^n \frac{x^k}{k!},$$

$$e^x < \frac{\sum_{k=0}^n \frac{x^k}{k!}}{1 - \frac{x^{n+1}}{(n+1)!}}.$$

Both inequalities now become

$$\sum_{k=0}^{n+1} \frac{x^k}{k!} < e^x < \frac{\sum_{k=0}^n \frac{x^k}{k!}}{1 - \frac{x^{n+1}}{(n+1)!}},$$

if x is positive

and if n is large enough to satisfy $x^{n+1} < (n+1)!$.

Example:

If x were to equal 1, and if n were to equal 5, this would give

$$2.718 < \frac{1957}{720} = \sum_{k=0}^6 \frac{1}{k!} < e < \frac{\sum_{k=0}^5 \frac{1}{k!}}{1 - \frac{1}{6!}} = \frac{1956}{719} < 2.7204.$$

On the other hand, if $x < 0$,

$$\begin{aligned} &\text{then } x < c < 0 \\ &\text{so that } 0 < e^x < f^{(n+1)}(c) = e^c < e^0 = 1. \end{aligned}$$

The Remainder $f^{(n+1)}(c) \frac{x^{n+1}}{(n+1)!}$ lies between 0 and $1 \cdot \frac{x^{n+1}}{(n+1)!}$

and $e^x = \sum_{k=0}^n \frac{x^k}{k!} + f^{(n+1)}(c) \frac{x^{n+1}}{(n+1)!}$ itself lies between

$$\sum_{k=0}^n \frac{x^k}{k!} + 0 \quad \text{and} \quad \sum_{k=0}^n \frac{x^k}{k!} + \frac{x^{n+1}}{(n+1)!}.$$

Thus $e^{\text{negative } x}$

lies between its consecutive partial Taylor Sums $\sum_{k=0}^n \frac{x^k}{k!}$ and $\sum_{k=0}^{n+1} \frac{x^k}{k!}$,

as in the cases of the ultimately alternating series for the sine and cosine.

Example: Let $f(x)$ equal $\ln x$ and let a equal 1.

$$f(x) = \ln x. \qquad f(1) = \ln 1 = 0.$$

$$f'(x) = \frac{1}{x}. \qquad f'(1) = \frac{1}{1} = 1.$$

$$f''(x) = -\frac{1}{x^2}. \qquad f''(1) = -1.$$

$$f^{(3)}(x) = \frac{2}{x^3}. \qquad f^{(3)}(1) = 2.$$

$$f^{(4)}(x) = -\frac{6}{x^4}. \qquad f^{(4)}(1) = -6.$$

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}. \qquad f^{(n)}(1) = (-1)^{n-1} (n-1)! .$$

$$\left(f^{(n+1)}(x) \text{ would equal } (-1)^{n-1} \frac{(n-1)!(-n)}{x^{n+1}} = (-1)^n \frac{(n)!}{x^{n+1}}. \right)$$

Taylor Polynomial of Degree n :

In x equals

$$\begin{aligned} 0 &+ 1 \frac{(x-1)^1}{1!} - 1 \frac{(x-1)^2}{2!} + 2 \frac{(x-1)^3}{3!} + \dots \\ &+ (-1)^{n-1} (n-1)! \frac{(x-1)^n}{n!} \\ &= \sum_{k=1}^n (-1)^{k-1} (k-1)! \frac{(x-1)^k}{k!} \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \dots + (-1)^{n-1} \frac{(x-1)^n}{n} \\ &= \sum_{k=1}^n (-1)^{k-1} \frac{(x-1)^k}{k} \end{aligned}$$

plus a remainder.

If $x > 1$, then we have $x > c > 1$ and $0 < \frac{1}{x^{n+1}} < \frac{1}{c^{n+1}} < 1$,
 so that the Lagrange Remainder

$$f^{(n+1)}(c) \frac{(x-1)^{n+1}}{(n+1)!} = (-1)^n \frac{n!}{c^{n+1}} \frac{(x-1)^{n+1}}{(n+1)!} = \frac{(-1)^n (x-1)^{n+1}}{c^{n+1} (n+1)}$$

lies between 0 and $(-1)^n \cdot 1 \cdot \frac{(x-1)^{n+1}}{(n+1)}$

so that $\ln x = \sum_{k=1}^n (-1)^{k-1} \frac{(x-1)^k}{k} + f^{(n+1)}(c) \frac{(x-1)^{n+1}}{(n+1)!}$

lies between $\sum_{k=1}^n (-1)^{k-1} \frac{(x-1)^k}{k} + 0$

and $\sum_{k=1}^n (-1)^{k-1} \frac{(x-1)^k}{k} + (-1)^n \frac{(x-1)^{n+1}}{(n+1)}$.

Thus, if $x > 1$ (and these will converge as $n \rightarrow \infty$ if $1 < x \leq 2$),

$\ln x$ lies between $\sum_{k=1}^n (-1)^{k-1} \frac{(x-1)^k}{k}$ and $\sum_{k=1}^{n+1} (-1)^{k-1} \frac{(x-1)^k}{k}$.

If $0 < x \leq 2$ then we can use the Mean Value Theorem for Integrals in a slightly different way on the Integral Remainder:

$$\begin{aligned}
 \int_{t=1}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt &= \int_{t=1}^x (-1)^n \frac{n!}{t^{n+1}} \frac{(x-t)^n}{n!} dt \\
 &= (-1)^{n-1} \int_{t=1}^x t \frac{(-1)(x-t)^n}{t^2 t^n} dt \\
 &= \frac{(-1)^{n-1}}{x} \int_{t=1}^x t \frac{(-x)}{t^2} \left(\frac{x-t}{t}\right)^n dt \\
 &= \frac{(-1)^{n-1}}{x} c \int_{t=1}^x \left(\frac{x-t}{t}\right)^n \left(\frac{-x}{t^2}\right) dt,
 \end{aligned}$$

where we have x constant and where we have c between 1 and x ,

$$\begin{aligned}
 &= \frac{(-1)^{n-1}}{x} c \int_{t=1}^x \left(\frac{x}{t} - 1\right)^n \frac{d}{dt} \left(\frac{x}{t} - 1\right) dt \\
 &= \frac{(-1)^{n-1}}{x} c \left[\frac{1}{n+1} \left(\frac{x}{t} - 1\right)^{n+1} \right]_{t=1}^x.
 \end{aligned}$$

If $0 < x \leq 2$ then the integral remainder $\int_{t=1}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$

equals
$$\frac{(-1)^{n-1}}{x} c \left[\frac{1}{n+1} \left(\frac{x}{t} - 1 \right)^{n+1} \right]_{t=1}^x$$

$$= \frac{(-1)^{n-1}}{x} c \left[\frac{1}{n+1} \left(\frac{x}{x} - 1 \right)^{n+1} - \frac{1}{n+1} \left(\frac{x}{1} - 1 \right)^{n+1} \right]$$

$$= \frac{(-1)^n}{x} c \frac{(x-1)^{n+1}}{n+1}, \text{ with } c \text{ between } 1 \text{ and } x;$$

so it lies between $(-1)^n \frac{(x-1)^{n+1}}{(n+1)x}$ and $(-1)^n \frac{(x-1)^{n+1}}{n+1}$,

and so that $\ln x$ lies between

$$\sum_{k=1}^n (-1)^{k-1} \frac{(x-1)^k}{k} + (-1)^n \frac{(x-1)^{n+1}}{(n+1)x}$$

and
$$\sum_{k=1}^n (-1)^{k-1} \frac{(x-1)^k}{k} + (-1)^n \frac{(x-1)^{n+1}}{n+1} = \sum_{k=1}^{n+1} (-1)^{k-1} \frac{(x-1)^k}{k} .$$

Each of the above sums $\sum_{k=0}^n$ will approach its limit $\sum_{k=0}^{\infty}$,
as n approaches ∞ ,

as their respective integral/Lagrange remainders approach 0.

We will get:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad \text{for all real } x.$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad \text{for all real } x.$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all real } x.$$

$$\ln x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k} \quad \text{for } 0 < x \leq 2.$$