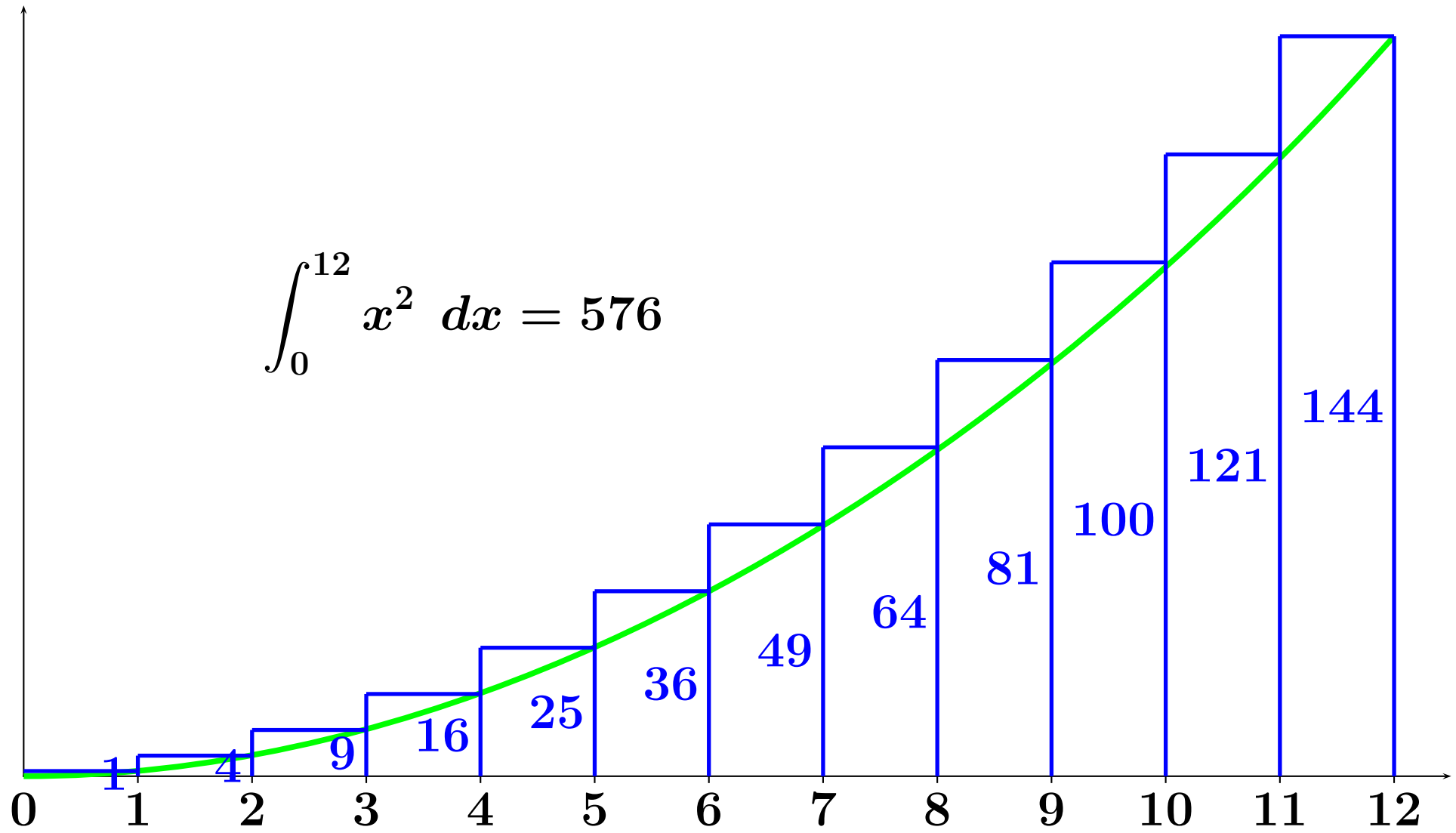
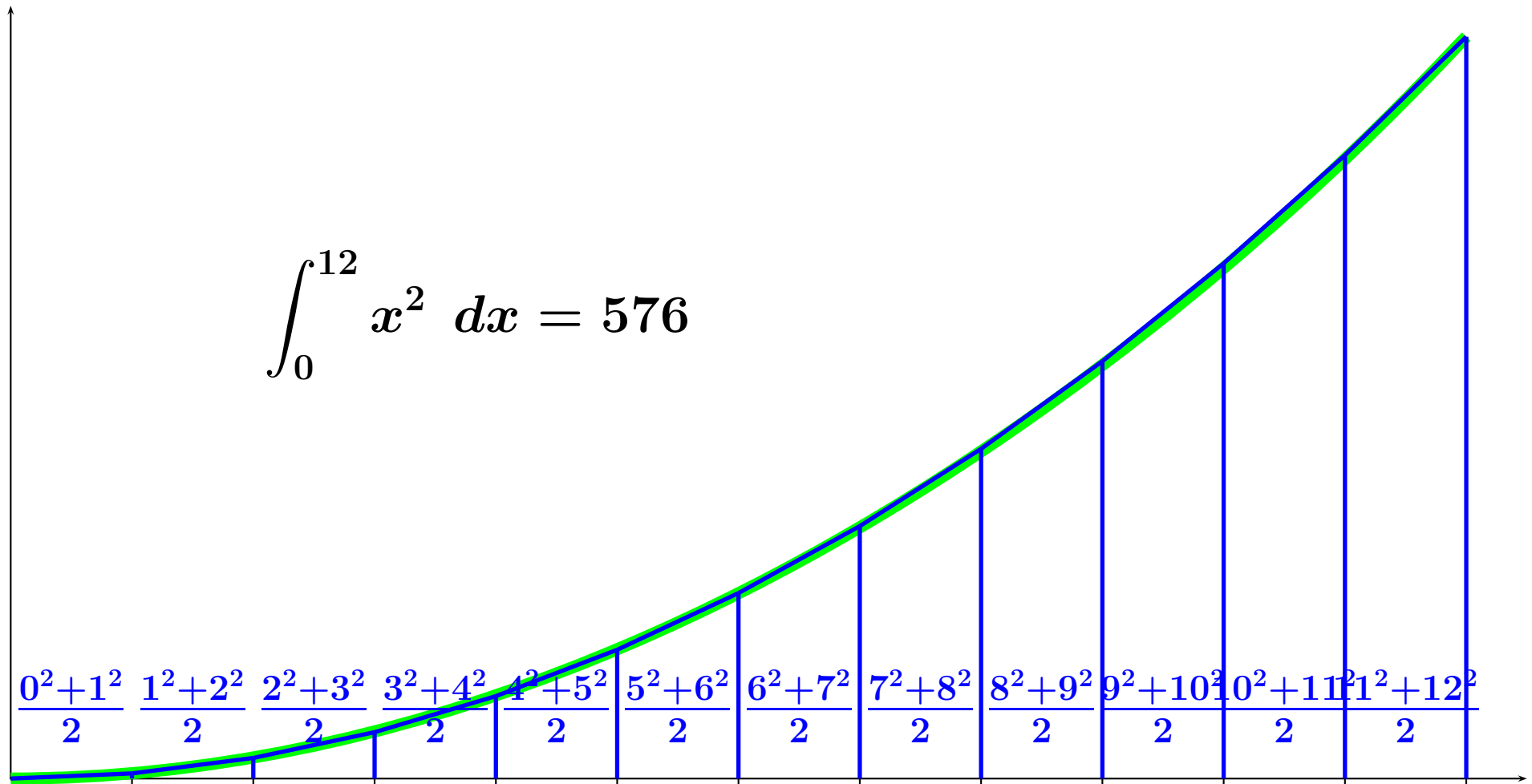


The Left Endpoint Sum equals 506.



The Right Endpoint Sum equals 650.

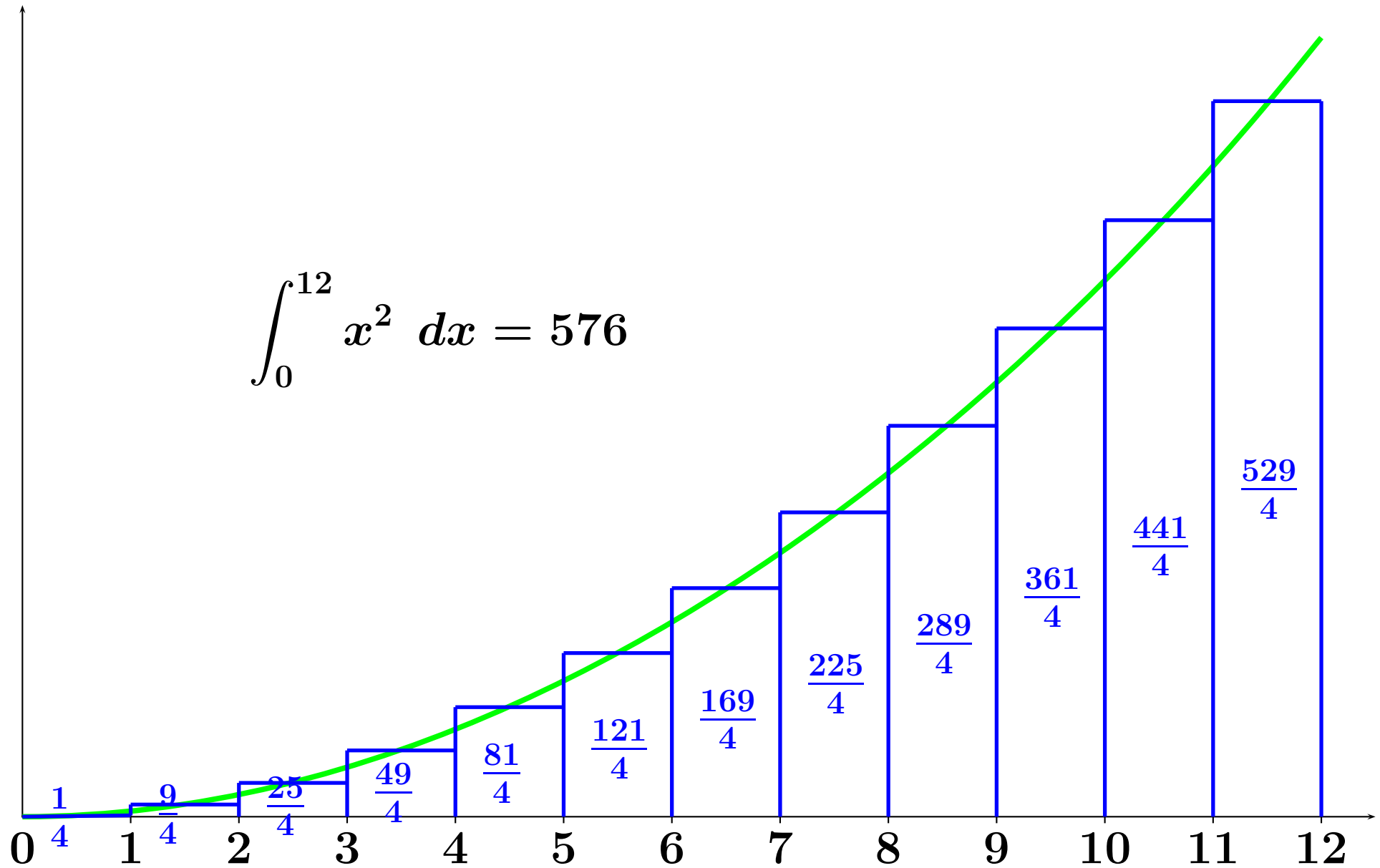


$$\int_0^{12} x^2 dx = 576$$

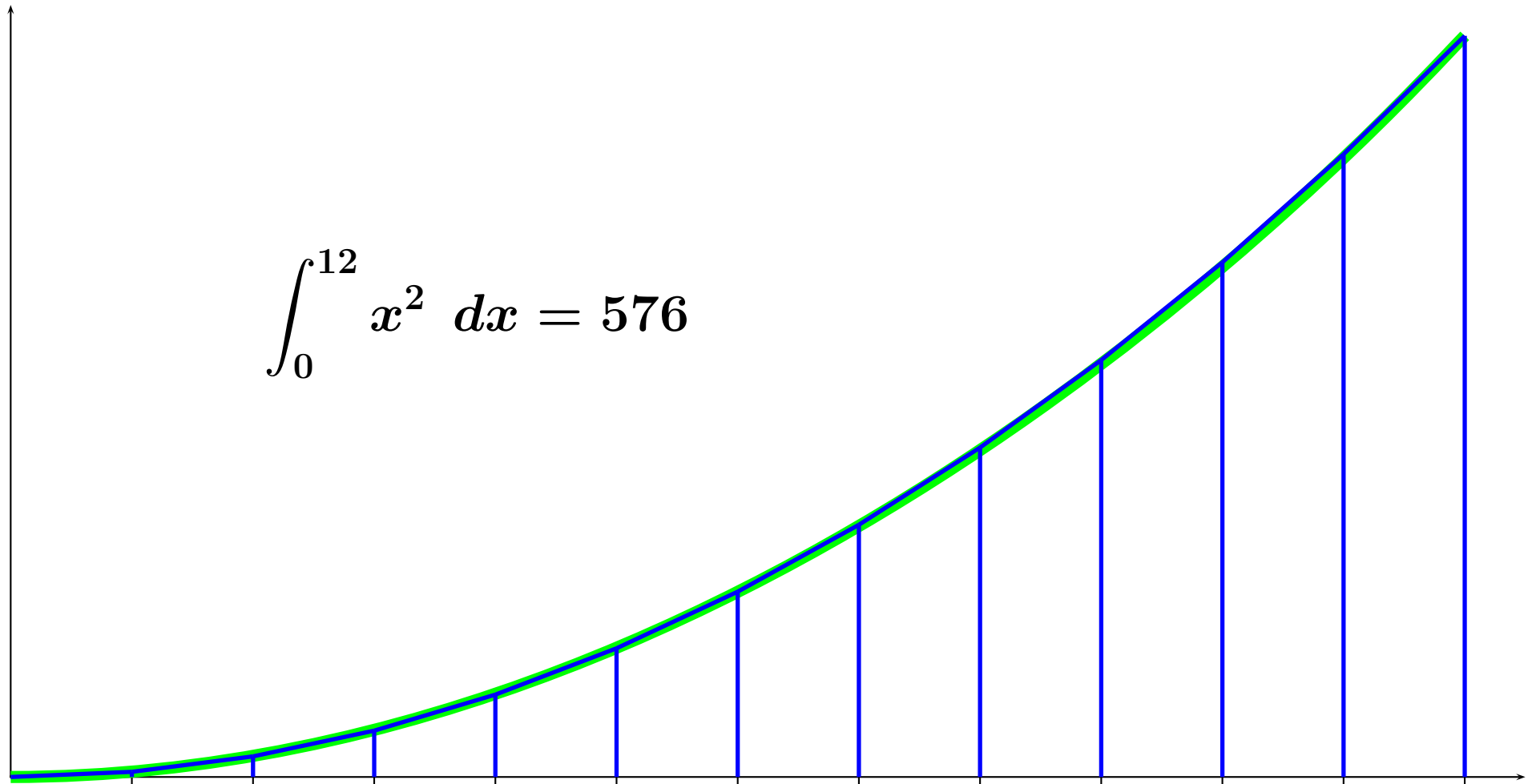
$$\frac{0^2+1^2}{2} \quad \frac{1^2+2^2}{2} \quad \frac{2^2+3^2}{2} \quad \frac{3^2+4^2}{2} \quad \frac{4^2+5^2}{2} \quad \frac{5^2+6^2}{2} \quad \frac{6^2+7^2}{2} \quad \frac{7^2+8^2}{2} \quad \frac{8^2+9^2}{2} \quad \frac{9^2+10^2}{2} \quad \frac{10^2+11^2}{2} \quad \frac{11^2+12^2}{2}$$

The Trapezoidal Sum equals

$$\frac{0^2}{2} + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + \frac{12^2}{2} = 578$$



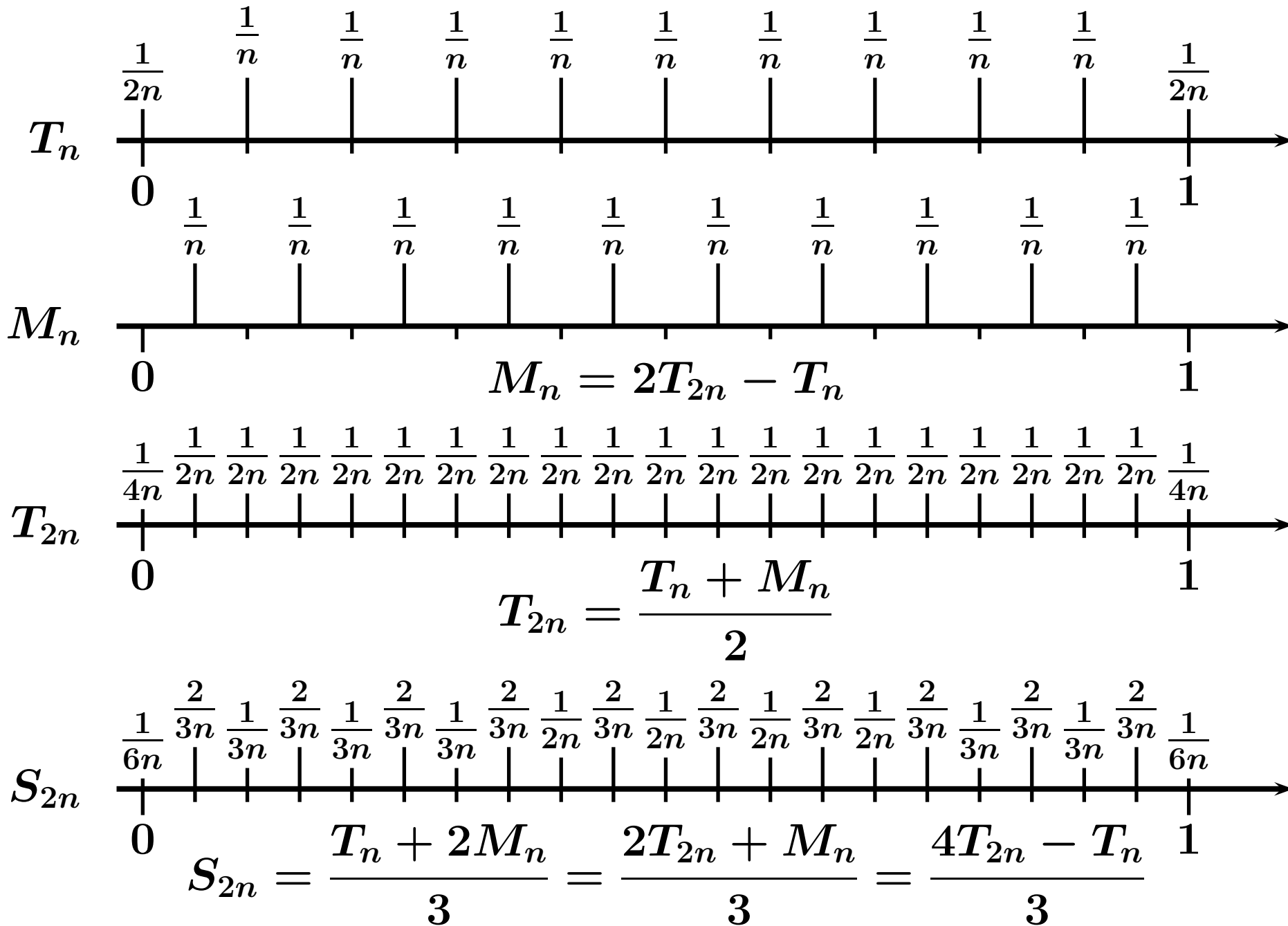
The Midpoint Sum equals 575.



$$\int_0^{12} x^2 dx = 576$$

The Simpson's (Parabolic) Sum equals

$$\frac{1}{3} (1 \cdot 0^2 + 4 \cdot 1^2 + 2 \cdot 2^2 + 4 \cdot 3^2 + 2 \cdot 4^2 + 4 \cdot 5^2 + 2 \cdot 6^2 + 4 \cdot 7^2 + 2 \cdot 8^2 + 4 \cdot 9^2 + 2 \cdot 10^2 + 4 \cdot 11^2 + 1 \cdot 12^2) = 576 \text{ (Exact)}$$



The Trapezoidal Integral Approximation Error is a sum of even powers of $\frac{1}{n}$.

$$\int_a^b f(x) dx = T_n + \frac{A}{n^2} + \frac{B}{n^4} + \frac{C}{n^6} + \dots$$

$$\int_a^b f(x) dx = T_{2n} + \frac{A}{(2n)^2} + \frac{B}{(2n)^4} + \frac{C}{(2n)^6} + \dots$$

$$\int_a^b f(x) dx = T_{2n} + \frac{A}{4n^2} + \frac{B}{16n^4} + \frac{C}{64n^6} + \dots$$

$$4 \int_a^b f(x) dx = 4T_{2n} + \frac{A}{n^2} + \frac{B}{4n^4} + \frac{C}{16n^6} + \dots$$

$$- \int_a^b f(x) dx = -T_n - \frac{A}{n^2} - \frac{B}{n^4} - \frac{C}{n^6} - \dots$$

$$3 \int_a^b f(x) dx = 4T_{2n} - T_n - \frac{3B}{4n^4} - \frac{15C}{16n^6} - \dots$$

$$\int_a^b f(x) dx = \frac{4T_{2n} - T_n}{3} - \frac{B}{4n^4} - \frac{5C}{16n^6} - \dots$$

Let $\frac{4T_{2n} - T_n}{3}$ be denoted by S_{2n} .

$$\begin{aligned}
 \int_a^b f(x) \, dx &= S_{2n} & + \frac{D}{n^4} & + \frac{E}{n^6} & + \dots \\
 \int_a^b f(x) \, dx &= S_{4n} & + \frac{D}{(2n)^4} & + \frac{E}{(2n)^6} & + \dots \\
 \int_a^b f(x) \, dx &= S_{4n} & + \frac{D}{16n^4} & + \frac{E}{64n^6} & + \dots \\
 16 \int_a^b f(x) \, dx &= 16S_{4n} & + \frac{D}{n^4} & + \frac{E}{4n^6} & + \dots \\
 - \int_a^b f(x) \, dx &= -S_{2n} & - \frac{D}{n^4} & - \frac{E}{n^6} & - \dots \\
 15 \int_a^b f(x) \, dx &= 16S_{4n} - S_{2n} & & - \frac{3E}{4n^6} & - \dots \\
 \int_a^b f(x) \, dx &= \frac{16S_{4n} - S_{2n}}{15} & & - \frac{E}{20n^6} & - \dots
 \end{aligned}$$

T_n Romberg Array, Starting with $T_n, T_{2n}, T_{4n}, T_{8n}, T_{16n}, T_{32n}$,

$$S_{2n} \equiv \frac{4T_{2n} - T_n}{3}$$

$$T_{2n} \quad R_{4n} \equiv \frac{16S_{4n} - S_{2n}}{15}$$

$$S_{4n} \equiv \frac{4T_{4n} - T_{2n}}{3} \quad Q_{8n} \equiv \frac{64R_{8n} - R_{4n}}{63}$$

$$T_{4n} \quad R_{8n} \equiv \frac{16S_{8n} - S_{4n}}{15} \quad P_{16n} \equiv \frac{256Q_{16n} - Q_{8n}}{255}$$

$$S_{8n} \equiv \frac{4T_{8n} - T_{4n}}{3} \quad Q_{16n} \equiv \frac{64R_{16n} - R_{8n}}{63} \quad U_{32n} \equiv \frac{1024P_{32n} - P_{16n}}{1023}$$

$$T_{8n} \quad R_{16n} \equiv \frac{16S_{16n} - S_{8n}}{15} \quad P_{32n} \equiv \frac{256Q_{32n} - Q_{16n}}{255}$$

$$S_{16n} \equiv \frac{4T_{16n} - T_{8n}}{3} \quad Q_{32n} \equiv \frac{64R_{32n} - R_{16n}}{63}$$

$$T_{16n} \quad R_{32n} \equiv \frac{16S_{32n} - S_{16n}}{15}$$

$$S_{32n} \equiv \frac{4T_{32n} - T_{16n}}{3}$$

T_{32n}

exact up to
 x^0, x^1

exact up to
 x^2, x^3

exact up to
 x^4, x^5

exact up to
 x^6, x^7

exact up to
 x^8, x^9

exact up to
 x^{10}, x^{11}

Romberg Array, Starting with $T_n, T_{2n}, T_{4n}, T_{8n}, T_{16n}, T_{32n},$

$$T_n \quad S_{2n} \equiv \frac{4T_{2n} - T_n}{3} \quad R_{4n} \equiv \frac{16S_{4n} - S_{2n}}{15} \quad Q_{8n} \equiv \frac{64R_{8n} - R_{4n}}{63} \quad P_{16n} \equiv \frac{256Q_{16n} - Q_{8n}}{255} \quad U_{32n} \equiv \frac{1024P_{32n} - P_{16n}}{1023}$$

$$T_{2n} \quad S_{4n} \equiv \frac{4T_{4n} - T_{2n}}{3} \quad R_{8n} \equiv \frac{16S_{8n} - S_{4n}}{15} \quad Q_{16n} \equiv \frac{64R_{16n} - R_{8n}}{63} \quad P_{32n} \equiv \frac{256Q_{32n} - Q_{16n}}{255}$$

$$T_{4n} \quad S_{8n} \equiv \frac{4T_{8n} - T_{4n}}{3} \quad R_{16n} \equiv \frac{16S_{16n} - S_{8n}}{15} \quad Q_{32n} \equiv \frac{64R_{32n} - R_{16n}}{63}$$

$$T_{8n} \quad S_{16n} \equiv \frac{4T_{16n} - T_{8n}}{3} \quad R_{32n} \equiv \frac{16S_{32n} - S_{16n}}{15}$$

$$T_{16n} \quad S_{32n} \equiv \frac{4T_{32n} - T_{16n}}{3}$$

T_{32n}

exact up to
 x^0, x^1

exact up to
 x^2, x^3

exact up to
 x^4, x^5

exact up to
 x^6, x^7

exact up to
 x^8, x^9

exact up to
 x^{10}, x^{11}

To Approximate $\int_1^2 \frac{1}{x} dx = \ln 2 = 0.693147181\dots$

$$T_1 = \frac{1}{1} \left(\frac{1}{2} \cdot \frac{1}{1} + \frac{1}{2} \cdot \frac{1}{2} \right) = 0.750000000$$

$$T_2 = \frac{1}{2} \left(\frac{1}{2} \cdot \frac{1}{1} + \frac{1}{\frac{3}{2}} + \frac{1}{2} \cdot \frac{1}{2} \right) = 0.708333333$$

$$T_4 = \frac{1}{4} \left(\frac{1}{2} \cdot \frac{1}{1} + \frac{1}{\frac{5}{4}} + \frac{1}{\frac{3}{2}} + \frac{1}{\frac{7}{4}} + \frac{1}{2} \cdot \frac{1}{2} \right) = 0.697023810$$

$$T_8 = \frac{1}{8} \left(\frac{1}{2} \cdot \frac{1}{1} + \frac{1}{\frac{9}{8}} + \frac{1}{\frac{5}{4}} + \frac{1}{\frac{11}{8}} + \frac{1}{\frac{3}{2}} + \frac{1}{\frac{13}{8}} + \frac{1}{\frac{7}{4}} + \frac{1}{\frac{15}{8}} + \frac{1}{2} \cdot \frac{1}{2} \right) = 0.694121850$$

To Approximate $\int_1^2 \frac{1}{x} dx = \ln 2 = 0.693147181\dots$

Romberg Array (Trapezoid)

0.750000000 0.694444444 0.693174604 0.693147477

0.708333333 0.693253969 0.693147901

0.697023810 0.693154530

0.694121850

$(4T_{2n} - T_n)/3 = (16S_{4n} - S_{2n})/15 = (64R_{8n} - R_{4n})/63 =$

Like the Trapezoid Method,
the Midpoint Method also has an error
which is a sum of even powers of $\frac{1}{n}$.

Since it uses only evaluation points
within the interior of the integration interval,
it is very handy for approximating improper integrals
like $\int_0^1 \frac{1}{\sqrt{x}} dx$.

To Approximate $\int_1^2 \frac{1}{x} dx = \ln 2 = 0.693147181\dots$

$$M_1 = \frac{1}{1} \left(\frac{1}{\frac{3}{2}} \right) = 0.666666666$$

$$M_2 = \frac{1}{2} \left(\frac{1}{\frac{5}{4}} + \frac{1}{\frac{7}{4}} \right) = 0.685714286$$

$$M_4 = \frac{1}{4} \left(\frac{1}{\frac{9}{8}} + \frac{1}{\frac{11}{8}} + \frac{1}{\frac{13}{8}} + \frac{1}{\frac{15}{8}} \right) = 0.691219891$$

$$M_8 = \frac{1}{8} \left(\frac{1}{\frac{17}{16}} + \frac{1}{\frac{19}{16}} + \frac{1}{\frac{21}{16}} + \frac{1}{\frac{23}{16}} + \frac{1}{\frac{25}{16}} + \frac{1}{\frac{27}{16}} + \frac{1}{\frac{29}{16}} + \frac{1}{\frac{31}{16}} \right) = 0.692660554$$

To Approximate $\int_1^2 \frac{1}{x} dx = \ln 2 = 0.693147181\dots$

Romberg Array (Midpoint)

0.6666666666 0.692063493 0.693121199 0.693148173

0.685714286 0.693055093 0.693146487

0.691219891 0.693140775

0.692660554

$$\begin{aligned} (4T_{2n} - T_n)/3 &= (16S_{4n} - S_{2n})/15 = (64R_{8n} - R_{4n})/63 = \end{aligned}$$

Instead of working only with $T_{2^k n}$ or with $M_{2^k n}$,
we can also use some (larger) divisors of any integer.

T_n

$$S_{cn} \equiv \frac{c^2 T_{cn} - T_n}{c^2 - 1}$$

T_{cn}

$$R_{cdn} \equiv \frac{d^4 S_{cdn} - S_{cn}}{d^4 - 1}$$

$$S_{cdn} \equiv \frac{d^2 T_{cdn} - T_{cn}}{d^2 - 1}$$

$$Q_{cden} \equiv \frac{e^6 R_{cden} - R_{cdn}}{e^6 - 1}$$

T_{cdn}

$$R_{cden} \equiv \frac{e^4 S_{cden} - S_{cdn}}{e^4 - 1}$$

$$S_{8n} \equiv \frac{e^2 T_{cden} - T_{cdn}}{e^2 - 1}$$

T_{cden}

exact up to
 x^0, x^1

exact up to
 x^2, x^3

exact up to
 x^4, x^5

exact up to
 x^6, x^7