How Close is an Integral to a Left Endpoint Rectangle's Area?

$$\int_a^b f(x)\ dx \ - \ f(a)(b-a)$$

$$= \int_a^b f(x)(x-b)'\ dx \ - \ f(a)(b-a)$$

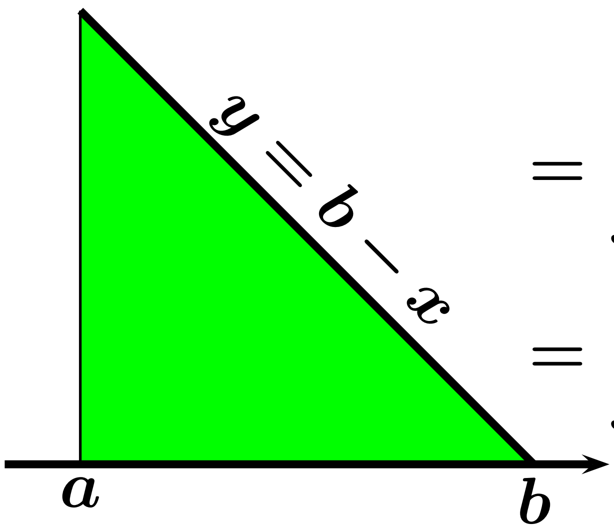
$$= f(x)(x-b)\Big|_a^b - \int_a^b f'(x)(x-b)\ dx \ - \ f(a)(b-a)$$

$$= f(b)(b-b) - f(a)(a-b)$$

$$+ \int_a^b f'(x)(b-x)\ dx \ - \ f(a)(b-a)$$

$$= \int_a^b f'(x)(b-x)\ dx$$

$$= \int_a^b f'(x)(b-a)\left(\frac{1}{2} - B_1\left(\frac{x-a}{b-a}\right)\right)\ dx$$

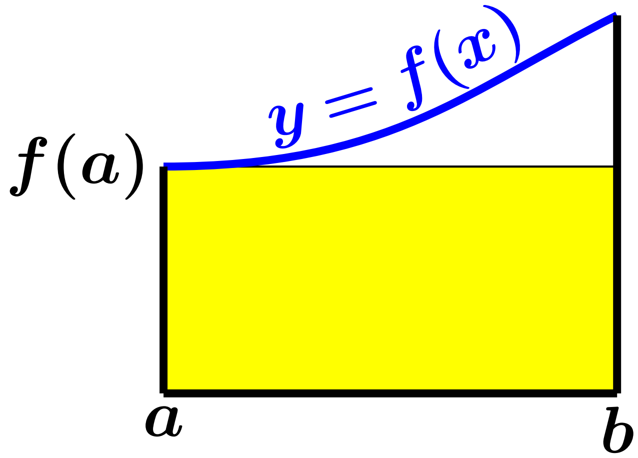


What does

$$\int_a^b f(x)\ dx - f(a)(b-a) = \int_a^b f'(x)(b-x)\ dx \quad \textbf{mean?}$$

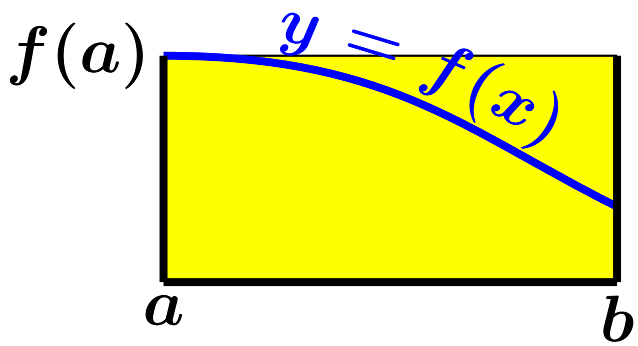
If $f'(x) > 0$,

then $\int_a^b f(x)\ dx > f(a)(b-a)$.



If $f'(x) < 0$,

then $\int_a^b f(x)\ dx < f(a)(b-a)$.



What else does

$$\int_a^b f(x)\ dx \ - \ f(a)(b-a) \ = \ \int_a^b f'(x)(b-x)\ dx \quad \textbf{mean?}$$

$$\int_a^b f(x)\ dx \ - \ f(a)(b-a) \ = \ f'(c)\int_a^b (b-x)\ dx \ \ \textbf{(IMVT)}$$

$$\int_a^b f(x)\ dx \ - \ f(a)(b-a) \ = \ f'(c)\frac{(b-a)^2}{2} \ \ (a \le c \le b)$$

If $|f'(x)| \le L$,

then $\left| \int_a^b f(x)\ dx \ - \ f(a)(b-a) \right| \le L\dfrac{(b-a)^2}{2}$.

If $f'(x) < 0$, **then** $\int_a^b f(x)\ dx \ < \ f(a)(b-a)$.

If $f'(x) > 0$, **then** $\int_a^b f(x)\ dx \ > \ f(a)(b-a)$.

How close is an Integral to a Left Rectangle Sum?

If the interval (a, b) is divided evenly by

$$a = x_0 \ < \ x_1 \ < \ x_2 \ < \ \dots \ < \ x_{n-1} \ < \ x_n = b,$$

we have $\int_a^b f(x)\ dx$ **minus the Left-Hand Sum**

$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\ dx \quad - \sum_{i=1}^{n} f(x_{i-1})(x_n - x_{n-1})$$

$$= \sum_{i=1}^{n} \left[\int_{x_{i-1}}^{x_i} f(x)\ dx - \qquad f(x_{i-1})(x_n - x_{n-1}) \right]$$

$$= \sum_{i=1}^{n} \left[\int_{x_{i-1}}^{x_i} f'(x)(x_i - x)\ dx \right]$$

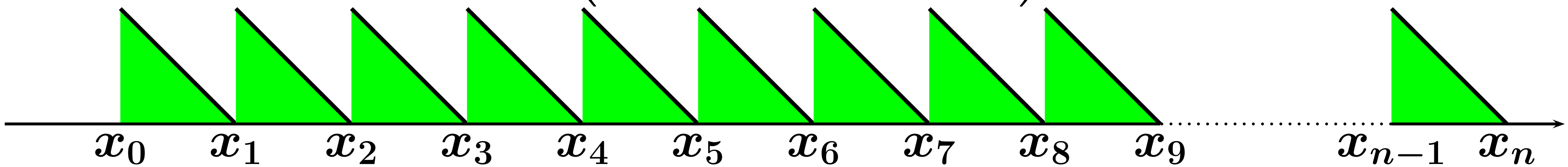
How close is an Integral to a Left Rectangle Sum?

$$\sum_{i=1}^{n}\left[\int_{x_{i-1}}^{x_i} f'(x)(x_i - x)\ dx\right] = \int_{x_0}^{x_n} f'(x)K_L(x)\ dx$$

$$= \int_a^b f'(x)K_L(x)dx = \int_a^b f'(x)\frac{b-a}{n}\left(\frac{1}{2}-\hat{B}_1\left(n\frac{x-a}{b-a}\right)\right)dx,$$

$$= f'(c)\int_a^b \frac{b-a}{n}\left(\frac{1}{2}-\hat{B}_1\left(n\frac{x-a}{b-a}\right)\right)dx = f'(c)\frac{(b-a)^2}{2n},$$

where $K_L = \dfrac{b-a}{n}\left(\dfrac{1}{2}-\hat{B}_1\left(n\dfrac{x-a}{b-a}\right)\right)$ is graphed by



and where $\displaystyle\int_a^b K_L(x)\ dx \quad = \quad n\frac{1}{2}\frac{(b-a)}{n}\frac{(b-a)}{n} \quad = \quad \frac{(b-a)^2}{2n}.$

$$\int_a^b f(x)\ dx \text{ minus its left endpoint sum equals } f'(c)\frac{(b-a)^2}{2n}.$$

If $|f'(x)| \leq L$,

then $$\left| \int_a^b f(x)\ dx - \text{its left endpoint sum} \right| \leq L\frac{(b-a)^2}{2n}.$$

If $f'(x) < 0$, then $\int_a^b f(x)\ dx <$ its left endpoint sum, which is really just an upper Darboux sum.

If $f'(x) > 0$, then $\int_a^b f(x)\ dx >$ its left endpoint sum, which is really just a lower Darboux sum.

How Close is an Integral to a Right Endpoint Rectangle?

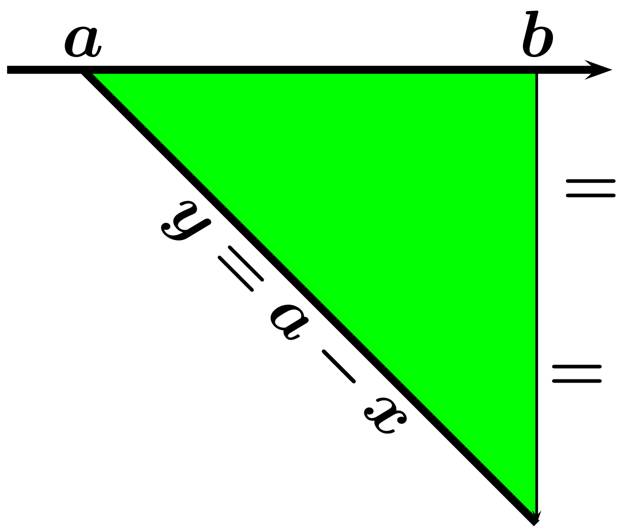
$$\int_a^b f(x)\ dx \ - \ f(b)(b-a)$$

$$= \int_a^b f(x)(x-a)'\ dx \ - \ f(b)(b-a)$$

$$= f(x)(x-a)\Big|_a^b - \int_a^b f'(x)(x-a)\ dx \ - \ f(b)(b-a)$$

$$= f(b)(b-a) - f(a)(a-a)$$

$$+ \int_a^b f'(x)(a-x)\ dx \ - \ f(b)(b-a)$$



$$= \int_a^b f'(x)(a-x)\ dx$$

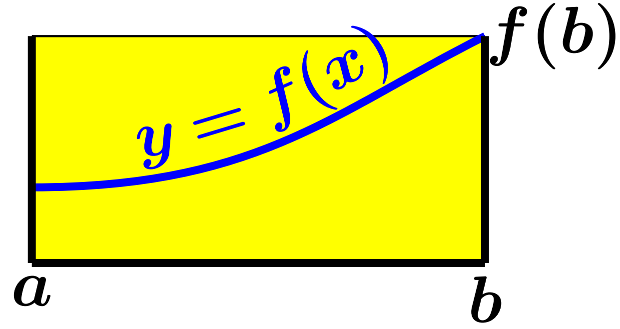
$$= \int_a^b f'(x)(b-a)\left(-\frac{1}{2} - B_1\left(\frac{x-a}{b-a}\right)\right)\ dx$$

What does

$$\int_a^b f(x)\ dx - f(b)(b-a) = \int_a^b f'(x)(a-x)\ dx \quad \text{mean?}$$

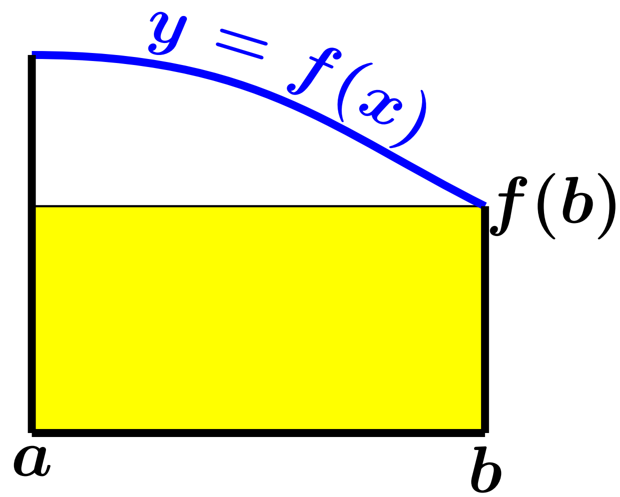
If $f'(x) > 0$,

then $\int_a^b f(x)\ dx < f(b)(b-a)$.



If $f'(x) < 0$,

then $\int_a^b f(x)\ dx > f(b)(b-a)$.



What else does

$$\int_a^b f(x)\ dx \ - \ f(b)(b-a) \ = \ \int_a^b f'(x)(a-x)\ dx \quad \text{mean?}$$

$$\int_a^b f(x)\ dx \ - \ f(b)(b-a) \ = \ f'(c)\int_a^b (a-x)\ dx \ \text{(IMVT)}$$

$$\int_a^b f(x)\ dx \ - \ f(b)(b-a) \ = \ -f'(c)\frac{(b-a)^2}{2} \ (a \le c \le b)$$

If $|f'(x)| \le L$,

$$\text{then} \left| \int_a^b f(x)\ dx \ - \ f(b)(b-a) \right| \le L\frac{(b-a)^2}{2}.$$

If $f'(x) < 0$, then $\int_a^b f(x)\ dx \ > \ f(b)(b-a)$.

If $f'(x) > 0$, then $\int_a^b f(x)\ dx \ < \ f(b)(b-a)$.

How close is an Integral to a Right Rectangle Sum?

If the interval (a, b) is divided evenly by

$$a = x_0 \;<\; x_1 \;<\; x_2 \;<\; \dots \;<\; x_{n-1} \;<\; x_n = b,$$

we have $\int_a^b f(x)\ dx$ **minus the Right-Hand Sum**

$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\ dx \quad - \sum_{i=1}^{n} f(x_i)(x_n - x_{n-1})$$

$$= \sum_{i=1}^{n} \left[\int_{x_{i-1}}^{x_i} f(x)\ dx \quad - \quad f(x_i)(x_n - x_{n-1}) \right]$$

$$= \sum_{i=1}^{n} \left[\int_{x_{i-1}}^{x_i} f'(x)(x_{i-1} - x)\ dx \right]$$

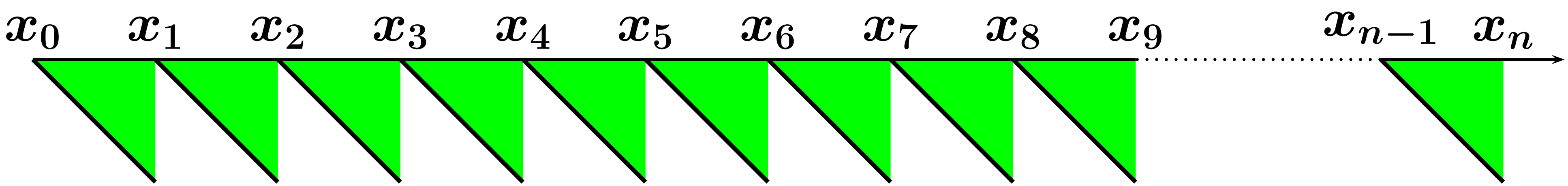
How close is an Integral to a Right Rectangle Sum?

$$= \sum_{i=1}^{n} \left[\int_{x_{i-1}}^{x_i} f'(x)(x_{i-1} - x) \; dx \right] = \int_{x_0}^{x_n} f'(x) K_R(x) \; dx$$

$$= \int_a^b f'(x) K_R(x) \; dx = f'(c) \int_a^b K_R(x) \; dx,$$

$$= f'(c) \int_a^b \frac{b-a}{n}\left(-\frac{1}{2} - \hat{B}_1\left(n \frac{x-a}{b-a} \right) \right) \; dx, = -f'(c) \frac{(b-a)^2}{2n},$$

where $K_R = \dfrac{b-a}{n}\left(-\dfrac{1}{2} - \hat{B}_1\left(n \dfrac{x-a}{b-a} \right) \right)$ is graphed by



where $\displaystyle\int_a^b K_R(x) dx = -n \frac{1}{2} \frac{(b-a)}{n} \frac{(b-a)}{n} = -\frac{(b-a)^2}{2n}.$

$$\int_a^b f(x)\ dx \ \text{ minus its left endpoint sum equals } -f'(c)\frac{(b-a)^2}{2n}.$$

If $|f'(x)| \leq L$,

$$\text{then} \left| \int_a^b f(x)\ dx - \text{ its right endpoint sum} \right| \leq L\frac{(b-a)^2}{2n}.$$

If $f'(x) > 0$, then $\int_a^b f(x)\ dx < $ its right endpoint sum, which is really just an upper Darboux sum.

If $f'(x) < 0$, then $\int_a^b f(x)\ dx > $ its right endpoint sum, which is really just a lower Darboux sum.

How Close is an Integral to the Area of a Single Trapezoid?

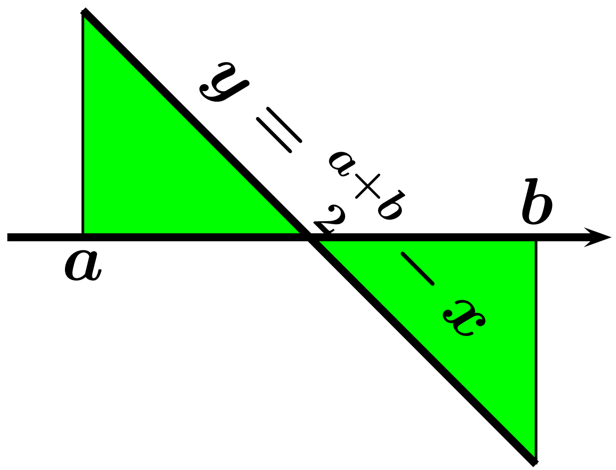
Recall the left-endpoint and the right-endpoint error formulas:

$$\int_a^b f(x)\ dx \ - \ f(a)(b-a) \ = \ \int_a^b f'(x)(b-x)\ dx,$$
$$\int_a^b f(x)\ dx \ - \ f(b)(b-a) \ = \ \int_a^b f'(x)(a-x)\ dx.$$

Average them:

$$\int_a^b f(x)dx - \frac{f(b)+f(a)}{2}(b-a) \ = \int_a^b f'(x)\left(\frac{a+b}{2} - x\right)dx.$$

The factor $\left(\frac{a+b}{2} - x\right)$ **changes sign.**



This would limit the usefulness of the Integral Mean-Value Theorem here.

To get a factor which doesn't change sign, integrate by parts:

$$\int_a^b f(x)dx - \frac{f(b)+f(a)}{2}(b-a) = \int_a^b f'(x)\left(\frac{a+b}{2} - x\right)dx$$

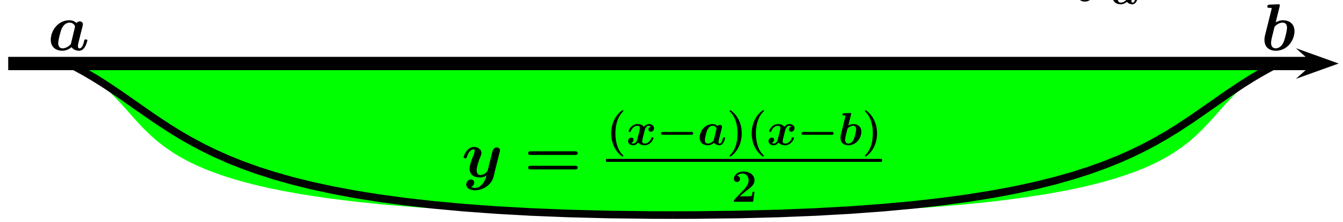
$$= -\frac{1}{2}\int_a^b f'(x)\big(2x-(a+b)\big)\ dx$$

$$= -\frac{1}{2}\int_a^b f'(x)\big(x^2-(a+b)x+ab\big)'\ dx$$

$$= -\frac{1}{2}\int_a^b f'(x)\big((x-a)(x-b)\big)'\ dx$$

$$= -\frac{1}{2}f'(x)(x-a)(x-b)\Big|_a^b + \frac{1}{2}\int_a^b f''(x)(x-a)(x-b)dx$$

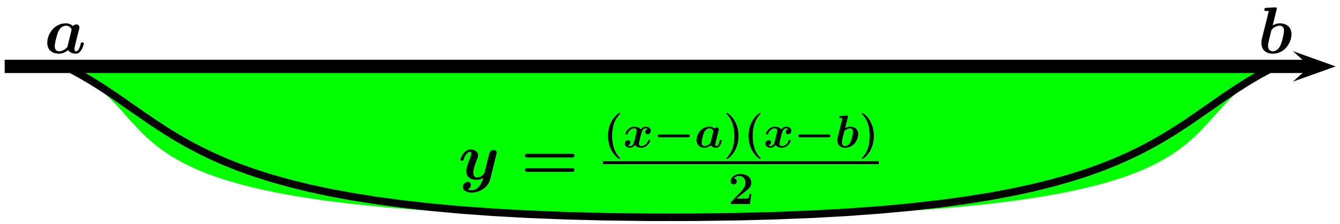
$$= \int_a^b f''(x)\frac{(x-a)(x-b)}{2}dx = f''(c)\int_a^b \frac{(x-a)(x-b)}{2}dx$$



An Integral minus the Area of a Single Trapezoid is equal to

$$\int_a^b f(x)dx - \frac{f(b)+f(a)}{2}(b-a) = \int_a^b f'(x)\left(\frac{a+b}{2} - x\right)dx$$

$$= \int_a^b f''(x)\frac{(x-a)(x-b)}{2}dx = f''(c)\int_a^b \frac{(x-a)(x-b)}{2}dx$$

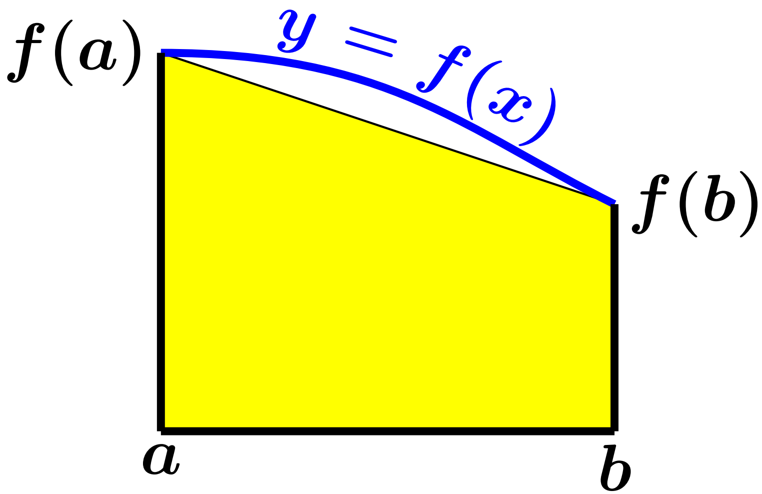


$$= -f''(c)\frac{(b-a)^3}{12}.$$

$$\int_a^b f(x)dx - \frac{f(b)+f(a)}{2}(b-a) = -f''(c)\frac{(b-a)^3}{12}.$$

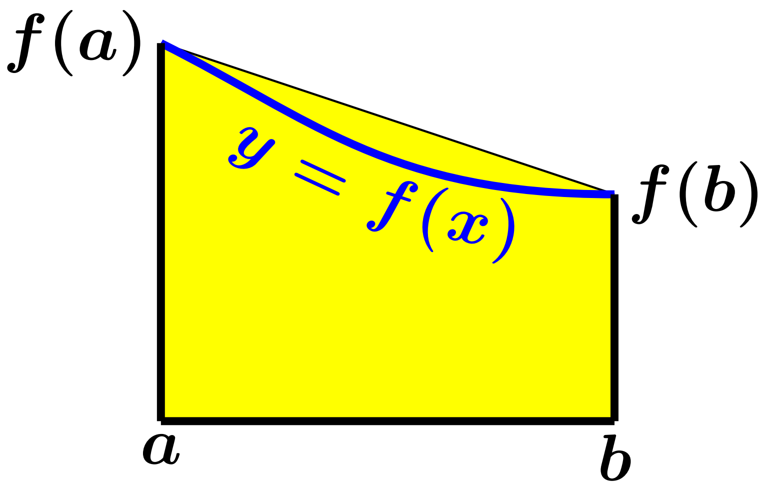
If $f''(x) < 0$ for all x, then

$$\int_a^b f(x)\ dx > \frac{f(b)+f(a)}{2}(b-a).$$



If $f''(x) > 0$ for all x, then

$$\int_a^b f(x)\ dx < \frac{f(b)+f(a)}{2}(b-a).$$



How close is an Integral to a Trapezoid Sum?

If the interval (a, b) is divided evenly by

$$a = x_0 \; < \; x_1 \; < \; x_2 \; < \; \dots \; < \; x_{n-1} \; < \; x_n = b,$$

then we have $\int_a^b f(x) \; dx$ minus the Trapezoid Sum

$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x) \; dx \; - \sum_{i=1}^{n} \frac{f(x_i) + f(x_{i-1})}{2}(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \left[\int_{x_{i-1}}^{x_i} f(x) \; dx \; - \frac{f(x_i) + f(x_{i-1})}{2}(x_i - x_{i-1}) \right]$$

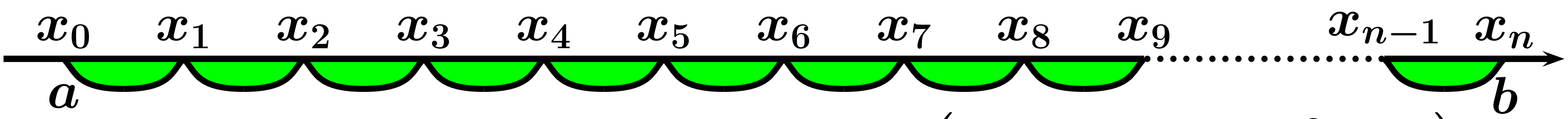
$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f''(x) \frac{(x - x_{i-1})(x - x_i)}{2} dx$$

$\int_a^b f(x)\ dx$ minus its trapezoidal sum equals

$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f''(x)\frac{(x-x_{i-1})(x-x_i)}{2}dx = \int_a^b f''(x)K_T(x)dx,$$

$$= f''(c)\int_a^b K_T(x)dx = -f''(c)\frac{(b-a)^3}{12n^2},$$

where $K_T(x)$ equals $\frac{(x-x_{i-1})(x-x_i)}{2}$ on each interval (x_{i-1}, x_i),



where $K_T(x)$ also equals $\frac{(b-a)^2}{2n^2}\left(\left\|n\frac{x-a}{b-a}+\frac{1}{2}\right\|^2-\frac{1}{4}\right)$,

and where $\int_a^b K_T(x)\ dx = -n\frac{1}{12}(x_i-x_{i-1})^3 = -\frac{n}{12}\left(\frac{b-a}{n}\right)^3$.

A Trapezoid sum is the average of Right and Left Sums,

which have kernals $$K_L = \frac{b-a}{n}\left(\frac{1}{2} - \hat{B}_1\left(n\frac{x-a}{b-a}\right)\right)$$

and $$K_R = \frac{b-a}{n}\left(-\frac{1}{2} - \hat{B}_1\left(n\frac{x-a}{b-a}\right)\right).$$

The average of these is $$-\frac{b-a}{n}\hat{B}_1\left(n\frac{x-a}{b-a}\right).$$

Since $$-\int_a^b f'(x)\frac{b-a}{n}\hat{B}_1\left(n\frac{x-a}{b-a}\right)dx$$

equals $$\int_a^b f''(x)\frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(n\frac{x-a}{b-a}\right) - B_2\right)dx,$$

the Trapezoid kernal, which goes with $f''(x)$, is equal to

$$K_T = \frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(n\frac{x-a}{b-a}\right) - B_2\right).$$

$\int_a^b f(x)\ dx$ **minus its trapezoidal sum equals** $-f''(c)\frac{(b-a)^3}{12n^2}$.

If $f(x)$ **is linear, i.e.**

if $f''(x) = 0$, **then** $\int_a^b f(x)\ dx \ = \$ **its trapezoidal sum.**

If $f''(x) < 0$, **then** $\int_a^b f(x)\ dx \ < \$ **its trapezoidal sum.**

If $f''(x) > 0$, **then** $\int_a^b f(x)\ dx \ > \$ **its trapezoidal sum.**

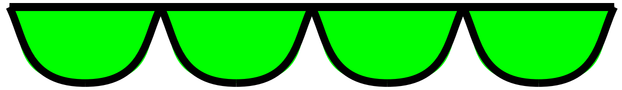
If $|f''(x)| \ \leq \ L$,

then $\left| \int_a^b f(x)\ dx \ - \ \text{its trapezoidal sum} \right| \ \leq \ L\frac{(b-a)^3}{12n^2}$.

$\int_a^b f(x)\ dx$ minus its n-Trapezoid sum equals $\int_a^b f''(x) K_{T_n}(x) dx$

where the n-Trapezoid kernal $K_{T_n}(x)$ is equal to

$$K_{T_n}(x) = \frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(n\frac{x-a}{b-a}\right) - B_2\right).$$



$\int_a^b f(x)\ dx$ minus its $2n$-Trapezoid sum equals $\int_a^b f''(x) K_{T_{2n}}(x) dx$

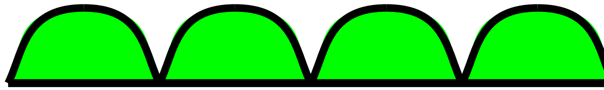
where the $2n$-Trapezoid kernal $K_{T_{2n}}(x)$ is equal to

$$K_{T_{2n}}(x) = \frac{(b-a)^2}{2(2n)^2}\left(\hat{B}_2\left(2n\frac{x-a}{b-a}\right) - B_2\right).$$

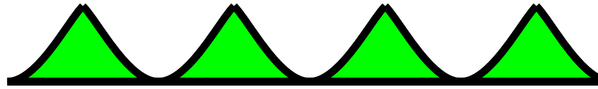
A midpoint sum M_n is equal to $2T_{2n} - T_n$.

$$2K_{T_{2n}}(x) = \frac{(b-a)^2}{4n^2}\left(\hat{B}_2\left(2n\frac{x-a}{b-a}\right) - B_2\right)$$

$$-K_{T_n}(x) = -\frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(n\frac{x-a}{b-a}\right) - B_2\right)$$



If we subtract these, $2K_{T_{2n}}(x) - K_{T_n}(x)$



we have the error kernal for the Midpoint method K_{M_n}:

$$K_{M_n}(x) = \frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(n\frac{x-a}{b-a} + \frac{1}{2}\right) + \frac{B_2}{2}\right)$$

Here is how that subtraction can take place:

$$\frac{(b-a)^2}{4n^2}\left(\hat{B}_2\left(2n\frac{x-a}{b-a}\right)-B_2\right)-\frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(n\frac{x-a}{b-a}\right)-B_2\right)$$
$$=\frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(n\frac{x-a}{b-a}+\frac{1}{2}\right)+\frac{B_2}{2}\right)$$

is true, **iff** $$\left(\hat{B}_2\left(2n\frac{x-a}{b-a}\right)-B_2\right)-2\left(\hat{B}_2\left(n\frac{x-a}{b-a}\right)-B_2\right)$$
$$=2\left(\hat{B}_2\left(n\frac{x-a}{b-a}+\frac{1}{2}\right)+\frac{B_2}{2}\right),$$

iff

$$\left(\hat{B}_2(2u)-B_2\right)-2\left(\hat{B}_2(u)-B_2\right) \;=\; 2\left(\hat{B}_2\left(u+\frac{1}{2}\right)+\frac{B_2}{2}\right),$$

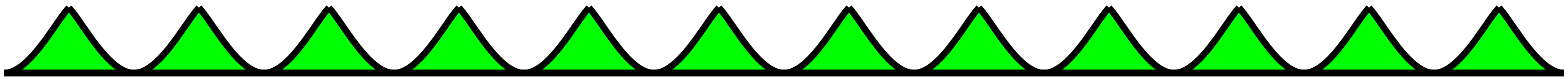
from the duplication formula: $\hat{B}_2(2u)-2\hat{B}_2(u)=2\hat{B}_2\left(u+\frac{1}{2}\right)$.

$\int_a^b f(x)\ dx$ minus its n-Midpoint sum equals

$$\int_a^b f''(x)K_{M_n}(x)dx = f''(c)\int_a^b K_{M_n}(x)dx = f''(c)\frac{(b-a)^3}{24n^2},$$

where the n-Midpoint kernal $K_{M_n}(x)$ is equal to

$$\frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(n\frac{x-a}{b-a}+\frac{1}{2}\right)+\frac{B_2}{2}\right) \;=\; \frac{(b-a)^2}{2n^2}\left\|n\frac{x-a}{b-a}\right\|^2.$$



and where its integral $\int_a^b K_{M_n}(x)dx$ is equal to

$$\int_a^b \frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(n\frac{x-a}{b-a}+\frac{1}{2}\right)+\frac{B_2}{2}\right)\ dx = (b-a)\frac{(b-a)^2}{2n^2}\ \frac{B_2}{2}.$$