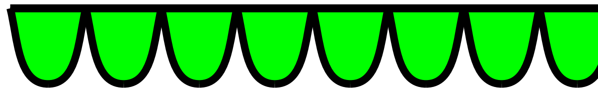
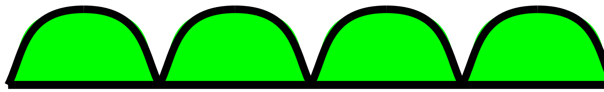
A Simpson's sum S_{2n} is equal to $\dfrac{4T_{2n} - T_n}{3}$.

If we subtract the corresponding error kernals,

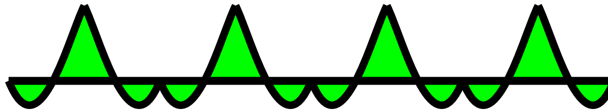
$$4K_{T_{2n}}(x) = \frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(2n\frac{x-a}{b-a}\right) - B_2\right)$$



$$-K_{T_n}(x) = -\frac{(b-a)^2}{2n^2}\left(\hat{B}_2\left(n\frac{x-a}{b-a}\right) - B_2\right)$$



we have a function which would change sign,



and which could not be an error kernal, itself.

However, the integral $\displaystyle\int_a^b f''(x)\frac{4K_{T_{2n}}(x) - K_{T_n}(x)}{3}\ dx$

can be integrated by parts twice to get a suitable error integral,

$\displaystyle\int_a^b f^{(4)}(x)K_{S_{2n}}(x)\ dx$, which has $K_{S_{2n}}(x) \le 0$.

The subtracted kernals $\dfrac{4K_{T_{2n}}(x) - K_{T_n}(x)}{3}$ **equal**

$$\frac{(b-a)^2}{2\cdot 3n^2}\left(\hat{B}_2\left(2n\frac{x-a}{b-a}\right) - B_2\left(n\frac{x-a}{b-a}\right)\right).$$

The second antiderivative $K_{S_{2n}}(x)$ **which** $= 0$ **at the endpoints equals** $\dfrac{(b-a)^4}{2\cdot 3\cdot 3\cdot 4n^4}\left(\dfrac{1}{4}\hat{B}_4\left(2n\dfrac{x-a}{b-a}\right) - \hat{B}_4\left(n\dfrac{x-a}{b-a}\right) + \dfrac{3}{4}B_4\right).$

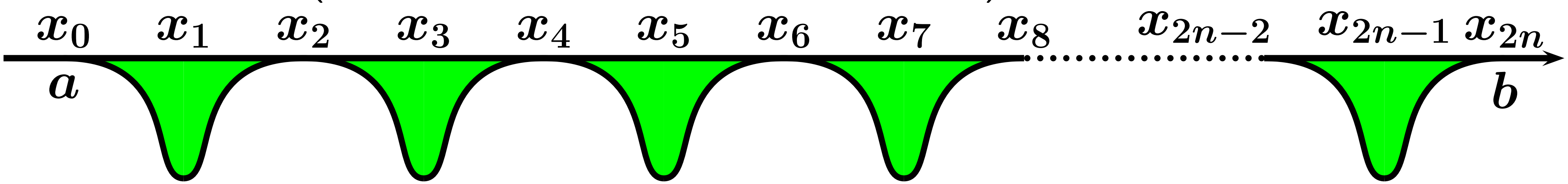
$$K_{S_{2n}}(x) = \frac{(b-a)^4}{72n^4}\left(\frac{8}{4}\hat{B}_4\left(n\frac{x-a}{b-a}\right) + \frac{8}{4}\hat{B}_4\left(n\frac{x-a}{b-a} + \frac{1}{2}\right)\right.$$
$$\left. - \hat{B}_4\left(n\frac{x-a}{b-a}\right) + \frac{3}{4}B_4\right),$$
$$= \frac{(b-a)^4}{72n^4}\left(\hat{B}_4\left(n\frac{x-a}{b-a}\right) + 2\hat{B}_4\left(n\frac{x-a}{b-a} + \frac{1}{2}\right) + \frac{3}{4}B_4\right).$$

The $2n-$Simpson kernal $K_{S_{2n}}(x)$ equals

$$\frac{(b-a)^4}{72n^4}\left(\hat{B}_4\left(n\frac{x-a}{b-a}\right)+2\hat{B}_4\left(n\frac{x-a}{b-a}+\frac{1}{2}\right)+\frac{3}{4}B_4\right)$$

$$=\frac{(b-a)^4}{72n^4}\left(B_4\left(\left\|n\frac{x-a}{b-a}\right\|\right)+2B_4\left(\left\|n\frac{x-a}{b-a}\right\|+\frac{1}{2}\right)+\frac{3}{4}B_4\right)$$

$$=\frac{(b-a)^4}{72n^4}\left(3\left\|n\frac{x-a}{b-a}\right\|^4-2\left\|n\frac{x-a}{b-a}\right\|^3\right)$$



$$\int_a^b K_{S_{2n}}(x)\ dx = \frac{(b-a)^4}{72n^4}(b-a)\left(0+0+\frac{3}{4}B_4\right)$$

$$=\frac{(b-a)^5}{96n^4}\left(-\frac{1}{30}\right) = -\frac{(b-a)^5}{2880n^4}.$$

The error integral $\int_a^b f^{(4)}(x) K_{S_{2n}}(x)\ dx$ equals

$f^{(4)}(c) \int_a^b K_{S_{2n}}(x)\ dx = -f^{(4)}(c) \frac{(b-a)^5}{2880 n^4}$., so that

$\int_a^b f(x)\ dx$ minus its $2n$-Simpson sum equals $-f^{(4)}(c) \frac{(b-a)^5}{180(2n)^4}$.

If $f(x)$ were a polynomial of degree three or less,

$f^{(4)}(x)$ would equal to zero for all x,

and $f^{(4)}(c) \frac{(b-a)^5}{180(2n)^4}$ would vanish.

The Simpson sum would equal the integral exactly.

Starting with the Euler-MacLauren Summation Formula

for $0 \le u \le n$,

For any integer n with $n > 0$ and any $s \ge 0$,

$$\sum_{i=0}^{n} g(i) = \int_{u=0}^{n} g(u)\ du + \frac{g(0)}{2} + \frac{g(n)}{2} + \sum_{\substack{j=2 \\ j\ \text{even}}}^{2s} \frac{B_j}{j!}\left(g^{(j-1)}(n) - g^{(j-1)}(0)\right) + \int_{u=0}^{n} g^{(2s+1)}(u)\frac{\hat{B}_{2s+1}(u)}{(2s+1)!}\ du$$

we shall replace u with $n\dfrac{x-a}{b-a}$, where $x = a + \dfrac{b-a}{n}u$.

We shall also define $f(x)$ to equal $g(u) = g\left(n\dfrac{x-a}{b-a}\right)$.

We shall replace u with $n\frac{x-a}{b-a}$, where $x = a + \frac{b-a}{n}u$.
This will yield $du = \frac{n}{b-a}\ dx$ within integrals.

Define $f(x)$ to equal $g(u) = g\left(n\frac{x-a}{b-a}\right)$.
When u equals 0, x equals a with $g(0) = f(a)$.
When u equals n, x equals b with $g(n) = f(b)$.

Since $g'(u)$ equals $\frac{dg(u)}{du} = \frac{df(x)}{du} = \frac{df(x)}{dx}\ \frac{dx}{du} = f'(x)\ \frac{b-a}{n}$,
by the chain rule,

$$\text{we shall have } g^{(k)}(u) = \left(\frac{b-a}{n}\right)^k f^{(k)}(x).$$

$$\sum_{i=0}^{n} g(i) = \int_{u=0}^{n} g(u)\ du + \frac{g(0)}{2} + \frac{g(n)}{2}$$

$$+ \sum_{\substack{j=2 \\ j \text{ even}}}^{2s} \frac{B_j}{j!}\left(g^{(j-1)}(n) - g^{(j-1)}(0)\right) + \int_{u=0}^{n} g^{(2s+1)}(u)\frac{\hat{B}_{2s+1}(u)}{(2s+1)!}\ du$$

becomes, under the above substitutions,

$$\sum_{i=0}^{n} f\left(a + \frac{b-a}{n} i\right) = \int_{x=a}^{b} f(x)\frac{n}{b-a}\ dx + \frac{f(a)}{2} + \frac{f(b)}{2}$$

$$+ \sum_{\substack{j=2 \\ j \text{ even}}}^{2s} \frac{B_j}{j!}\left(\left(\frac{b-a}{n}\right)^{j-1} f^{(j-1)}(b) - \left(\frac{b-a}{n}\right)^{j-1} f^{(j-1)}(a)\right)$$

$$+ \int_{x=a}^{b} \left(\frac{b-a}{n}\right)^{2s+1} f^{(2s+1)}(x)\frac{\hat{B}_{2s+1}\left(n\frac{x-a}{b-a}\right)}{(2s+1)!}\ \frac{n}{b-a}\ dx$$

We now have the Euler-MacLauren Summation Formula

for $a \le x \le b$,

For any numbers a and b and any $s \ge 0$,

$$\sum_{i=0}^{n} f\left(a + \frac{b-a}{n} i\right) = \frac{n}{b-a} \int_{x=a}^{b} f(x)\ dx + \frac{f(a)}{2} + \frac{f(b)}{2}$$

$$+ \sum_{\substack{j=2 \\ j \text{ even}}}^{2s} \frac{B_j}{j!} \left(\frac{b-a}{n}\right)^{j-1} \left(f^{(j-1)}(b) - f^{(j-1)}(a)\right)$$

$$+ \left(\frac{b-a}{n}\right)^{2s} \int_{x=a}^{b} f^{(2s+1)}(x) \frac{\hat{B}_{2s+1}\left(n \frac{x-a}{b-a}\right)}{(2s+1)!}\ dx$$

Noting that $\dfrac{b-a}{n}\left(\sum_{i=0}^{n} f\left(a+\dfrac{b-a}{n}i\right) - \dfrac{f(a)}{2} - \dfrac{f(b)}{2}\right)$,

which equals $\dfrac{b-a}{n}\left(\dfrac{f(a)}{2} + \sum_{i=1}^{n-1} f\left(a+\dfrac{b-a}{n}i\right) + \dfrac{f(b)}{2}\right)$,

happens to be the n-Trapezoid sum, T_n **for** f **on** $[a,b]$**, we have**

For any numbers a **and** b **and any** $s \geq 0$**,**

$$T_n = \int_a^b f(x)\ dx + \sum_{\substack{j=2 \\ j \text{ even}}}^{2s} \frac{B_j}{j!}\left(\frac{b-a}{n}\right)^j \left(f^{(j-1)}(b) - f^{(j-1)}(a)\right) + \left(\frac{b-a}{n}\right)^{2s+1} \int_a^b f^{(2s+1)}(x) \frac{\hat{B}_{2s+1}\left(n\dfrac{x-a}{b-a}\right)}{(2s+1)!}\ dx$$

The last term, $\left(\frac{b-a}{n}\right)^{2s+1} \int_a^b f^{(2s+1)}(x) \frac{\hat{B}_{2s+1}\left(n\frac{x-a}{b-a}\right)}{(2s+1)!}\, dx,$

can become, under an integration by parts,

$$\left(\frac{b-a}{n}\right)^{2s+2} \int_a^b f^{(2s+1)}(x) d\left(\frac{\hat{B}_{2s+2}\left(n\frac{x-a}{b-a}\right) - B_{2s+2}}{(2s+2)!}\right)$$

$$= -\left(\frac{b-a}{n}\right)^{2s+2} \int_a^b f^{(2s+2)}(x) \frac{\hat{B}_{2s+2}\left(n\frac{x-a}{b-a}\right) - B_{2s+2}}{(2s+2)!} dx,$$

and occasionally under the integral mean value theorem,

$$= -\left(\frac{b-a}{n}\right)^{2s+2} f^{(2s+2)}(c) \int_a^b \frac{\hat{B}_{2s+2}\left(n\frac{x-a}{b-a}\right) - B_{2s+2}}{(2s+2)!} dx,$$

$$= \frac{(b-a)^{2s+3} B_{2s+2}}{(2s+2)!\, n^{2s+2}} f^{(2s+2)}(c).$$

$$
\begin{aligned}
T_n = & \int_a^b f(x)\ dx \\
& + \sum_{\substack{j=2 \\ j \text{ even}}}^{2s} \frac{B_j}{j!} \left(\frac{b-a}{n}\right)^j \left(f^{(j-1)}(b) - f^{(j-1)}(a)\right) \\
& - \left(\frac{b-a}{n}\right)^{2s+2} \int_a^b f^{(2s+2)}(x) \frac{\hat{B}_{2s+2}\left(n\frac{x-a}{b-a}\right) - B_{2s+2}}{(2s+2)!} dx
\end{aligned}
$$

agrees, if $s = 0$, with our earlier trapezoid remainder

$$
\begin{aligned}
T_n = & \int_a^b f(x)\ dx \\
& - \left(\frac{b-a}{n}\right)^2 \int_a^b f^{(2)}(x) \frac{\hat{B}_2\left(n\frac{x-a}{b-a}\right) - B_2}{2!} dx
\end{aligned}
$$

$$T_n = \int_a^b f(x)\ dx + \frac{B_2}{2!}\left(\frac{b-a}{n}\right)^2\Big(f'(b) - f'(a)\Big) + \sum_{\text{even } j=4}^{2s} \frac{B_j}{j!}\frac{(b-a)^j}{n^j}\Big(f^{(j-1)}(b) - f^{(j-1)}(a)\Big) - \frac{(b-a)^{2s+2}}{n^{2s+2}}\int_a^b f^{(2s+2)}(x)\frac{\hat{B}_{2s+2}\left(n\frac{x-a}{b-a}\right) - B_{2s+2}}{(2s+2)!}dx$$

$$T_{cn} = \int_a^b f(x)\ dx + \frac{B_2}{2!}\left(\frac{b-a}{cn}\right)^2\Big(f'(b) - f'(a)\Big) + \sum_{\text{even } j=4}^{2s} \frac{B_j}{j!}\frac{(b-a)^j}{(cn)^j}\Big(f^{(j-1)}(b) - f^{(j-1)}(a)\Big) - \frac{(b-a)^{2s+2}}{(cn)^{2s+2}}\int_a^b f^{(2s+2)}(x)\frac{\hat{B}_{2s+2}\left(cn\frac{x-a}{b-a}\right) - B_{2s+2}}{(2s+2)!}dx$$

$$T_n = \int_a^b f(x)\ dx + \frac{B_2}{2!}\left(\frac{b-a}{n}\right)^2\left(f'(b) - f'(a)\right)$$
$$+ \sum_{\text{even } j=4}^{2s} \frac{B_j}{j!}\frac{(b-a)^j}{n^j}\left(f^{(j-1)}(b) - f^{(j-1)}(a)\right)$$
$$- \frac{(b-a)^{2s+2}}{n^{2s+2}} \int_a^b f^{(2s+2)}(x)\frac{\hat{B}_{2s+2}\left(n\frac{x-a}{b-a}\right) - B_{2s+2}}{(2s+2)!}dx$$

$$c^2 T_{cn} = c^2 \int_a^b f(x)\ dx + c^2\frac{B_2}{2!}\left(\frac{b-a}{cn}\right)^2\left(f'(b) - f'(a)\right)$$
$$+ \sum_{\text{even } j=4}^{2s} \frac{B_j}{j!}\frac{c^2(b-a)^j}{c^j n^j}\left(f^{(j-1)}(b) - f^{(j-1)}(a)\right)$$
$$- \frac{(b-a)^{2s+2}}{c^{2s} n^{2s+2}} \int_a^b f^{(2s+2)}(x)\frac{\hat{B}_{2s+2}\left(cn\frac{x-a}{b-a}\right) - B_{2s+2}}{(2s+2)!}dx$$

$$
\begin{aligned}
c^2 T_{cn} - T_n = (c^2-1)\int_a^b f(x)\ dx + 0 \\
+ \sum_{\text{even } j=4}^{2s} \frac{B_j}{j!}\left(\frac{c^2}{c^j}-1\right)\frac{(b-a)^j}{n^j}\left(f^{(j-1)}(b)-f^{(j-1)}(a)\right) \\
- \frac{(b-a)^{2s+2}}{n^{2s+2}}\int_a^b \frac{f^{(2s+2)}(x)}{(2s+2)!}\ \cdot
\end{aligned}
$$

$$
\left(\frac{1}{c^{2s}}\hat{B}_{2s+2}\left(cn\frac{x-a}{b-a}\right)-\frac{1}{c^{2s}}B_{2s+2}-\hat{B}_{2s+2}\left(n\frac{x-a}{b-a}\right)+B_{2s+2}\right)dx,
$$

$$
\begin{aligned}
S_{cn} = \frac{c^2 T_{cn} - T_n}{c^2-1} = \int_a^b f(x)\ dx \\
+ \sum_{\text{even } j=4}^{2s} \frac{B_j}{j!}\left(\frac{c^2-c^j}{c^j(c^2-1)}\right)\frac{(b-a)^j}{n^j}\left(f^{(j-1)}(b)-f^{(j-1)}(a)\right) \\
- \frac{(b-a)^{2s+2}}{(c^2-1)n^{2s+2}}\int_a^b \frac{f^{(2s+2)}(x)}{(2s+2)!}\ \cdot\ \left(\text{(same kernal)}\right)dx
\end{aligned}
$$

If s equals 1, and if c equals 2, we have

$$S_{2n} = \frac{4T_{cn} - T_n}{4-1} = \int_a^b f(x)\ dx - \frac{(b-a)^4}{(4-1)n^4} \int_a^b \frac{f^{(4)}(x)}{4!} \cdot \left(\frac{1}{4}\hat{B}_4\left(cn\frac{x-a}{b-a}\right) - \frac{1}{4}B_4 - \hat{B}_4\left(n\frac{x-a}{b-a}\right) + B_4 \right) dx,$$

$$S_{2n} = \int_a^b f(x)\ dx - \frac{(b-a)^4}{72n^4} \int_a^b f^{(4)}(x) \left(3\left\| n\frac{x-a}{b-a} \right\|^4 - 2\left\| n\frac{x-a}{b-a} \right\|^3 \right) dx,$$

$$S_{2n} = \int_a^b f(x)\ dx - f^{(4)}(?)\frac{(b-a)^5}{180(2n)^4}.$$

This error formula for the Second Columns in Romberg Tables agrees with the error formula for the Simpson Approximation.

Now we go back to more general values of s and c:

$$S_{cn} = \frac{c^2 T_{cn} - T_n}{c^2 - 1} = \int_a^b f(x)\ dx$$

$$+ \frac{B_4}{4!} \frac{(c^2 - c^4)}{c^4(c^2 - 1)} \frac{(b-a)^4}{n^4} \Big(f^{(3)}(b) - f^{(3)}(a) \Big)$$

$$+ \sum_{\text{even } j=6}^{2s} \frac{B_j}{j!} \frac{(c^2 - c^j)}{c^j(c^2 - 1)} \frac{(b-a)^j}{n^j} \Big(f^{(j-1)}(b) - f^{(j-1)}(a) \Big)$$

$$- \frac{(b-a)^{2s+2}}{(c^2 - 1) n^{2s+2}} \int_a^b \frac{f^{(2s+2)}(x)}{(2s+2)!} \cdot \left(\frac{1}{c^{2s}} \hat{B}_{2s+2}\left(cn \frac{x-a}{b-a} \right) - \frac{1}{c^{2s}} B_{2s+2} - \hat{B}_{2s+2}\left(n \frac{x-a}{b-a} \right) + B_{2s+2} \right) dx,$$

We multiply by d^4 and replace each n by dn

$$d^4 S_{cdn} = \frac{d^4 c^2 T_{cdn} - d^4 T_{dn}}{c^2 - 1} = d^4 \int_a^b f(x)\ dx$$

$$+ \frac{B_4}{4!} \frac{(c^2 - c^4)}{c^4(c^2 - 1)} d^4 \frac{(b-a)^4}{d^4 n^4} \Big(f^{(3)}(b) - f^{(3)}(a) \Big)$$

$$+ \sum_{\text{even } j=6}^{2s} \frac{B_j}{j!} \frac{(c^2 - c^j)}{c^j(c^2 - 1)} d^4 \frac{(b-a)^j}{d^j n^j} \Big(f^{(j-1)}(b) - f^{(j-1)}(a) \Big)$$

$$- \frac{(b-a)^{2s+2} d^4}{(c^2-1) d^{2s+2} n^{2s+2}} \int_a^b \frac{f^{(2s+2)}(x)}{(2s+2)!} \cdot$$

$$\left(\frac{1}{c^{2s}} \hat{B}_{2s+2}\left(cdn \frac{x-a}{b-a} \right) - \frac{1}{c^{2s}} B_{2s+2} \right.$$

$$\left. - \hat{B}_{2s+2}\left(dn \frac{x-a}{b-a} \right) + B_{2s+2} \right) dx$$

Subtracting, we get:

$$d^4 S_{cdn} - S_{cn} \;=\; (d^4-1)\int_a^b f(x)\;dx \;+\; 0$$

$$+\sum_{\text{even } j=6}^{2s} \frac{B_j}{j!}\frac{(c^2-c^j)}{c^j(c^2-1)}\left(\frac{d^4}{d^j}-1\right)\frac{(b-a)^j}{n^j}\left(f^{(j-1)}(b)-f^{(j-1)}(a)\right)$$

$$-\;\frac{(b-a)^{2s+2}}{(c^2-1)n^{2s+2}}\int_a^b \frac{f^{(2s+2)}(x)}{(2s+2)!}\;\cdot$$

$$\Bigg(\frac{1}{d^{2s-2}c^{2s}}\hat{B}_{2s+2}\left(cdn\frac{x-a}{b-a}\right)-\frac{1}{d^{2s-2}c^{2s}}B_{2s+2}$$

$$-\;\frac{1}{d^{2s-2}}\hat{B}_{2s+2}\left(dn\frac{x-a}{b-a}\right)+\frac{1}{d^{2s-2}}B_{2s+2}$$

$$-\;\frac{1}{c^{2s}}\hat{B}_{2s+2}\left(cn\frac{x-a}{b-a}\right)+\frac{1}{c^{2s}}B_{2s+2}$$

$$+\;\hat{B}_{2s+2}\left(n\frac{x-a}{b-a}\right)-B_{2s+2}\Bigg)dx,$$

$$\frac{d^4 S_{cdn} - S_{cn}}{d^4-1} = \frac{d^4c^2T_{cdn} - d^4T_{dn} - c^2T_{cn} + T_n}{(d^4-1)(c^2-1)} = \int_a^b f(x)\ dx$$

$$+ \sum_{\text{even } j=6}^{2s} \frac{B_j}{j!}\frac{(c^2-c^j)}{c^j(c^2-1)}\frac{(d^4-d^j)}{d^j(d^4-1)}\frac{(b-a)^j}{n^j}\Big(f^{(j-1)}(b) - f^{(j-1)}(a)\Big)$$

$$- \frac{(b-a)^{2s+2}}{(c^2-1)(d^4-1)n^{2s+2}}\int_a^b \frac{f^{(2s+2)}(x)}{(2s+2)!}\ \cdot$$

$$\left(\frac{1}{d^{2s-2}c^{2s}}\hat{B}_{2s+2}\left(cdn\frac{x-a}{b-a}\right) - \frac{1}{d^{2s-2}c^{2s}}B_{2s+2} - \frac{1}{d^{2s-2}}\hat{B}_{2s+2}\left(dn\frac{x-a}{b-a}\right) + \frac{1}{d^{2s-2}}B_{2s+2} - \frac{1}{c^{2s}}\hat{B}_{2s+2}\left(cn\frac{x-a}{b-a}\right) + \frac{1}{c^{2s}}B_{2s+2} + \hat{B}_{2s+2}\left(n\frac{x-a}{b-a}\right) - B_{2s+2}\right)dx,$$

If s equals 2, we have

$$R_{cdn} = \frac{d^4 S_{cdn} - S_{cn}}{d^4 - 1} = \frac{d^4 c^2 T_{cdn} - d^4 T_{dn} - c^2 T_{cn} + T_n}{(d^4 - 1)(c^2 - 1)}$$

$$= \int_a^b f(x)\ dx$$

$$- \frac{(b-a)^6}{(c^2-1)(d^4-1)n^6} \int_a^b \frac{f^{(6)}(x)}{6!} \cdot \left(\frac{1}{d^2 c^4} \hat{B}_6\left(cdn\frac{x-a}{b-a}\right) - \frac{1}{d^2 c^4} B_6 - \frac{1}{d^2} \hat{B}_6\left(dn\frac{x-a}{b-a}\right) + \frac{1}{d^2} B_6 - \frac{1}{c^4} \hat{B}_6\left(cn\frac{x-a}{b-a}\right) + \frac{1}{c^4} B_6 + \hat{B}_6\left(n\frac{x-a}{b-a}\right) - B_6 \right) dx,$$

If s equals 2, we have

$$\boldsymbol{R_{cdn} = \frac{d^4 S_{cdn} - S_{cn}}{d^4 - 1} = \frac{d^4 c^2 T_{cdn} - d^4 T_{dn} - c^2 T_{cn} + T_n}{(d^4 - 1)(c^2 - 1)}}$$

$$\boldsymbol{= \int_a^b f(x)\ dx}$$

$$\boldsymbol{- \ \frac{(b-a)^6 f^{(6)}(?)}{6!(c^2-1)(d^4-1)n^6} \int_a^b \left(\frac{1}{d^2c^4} \hat{B}_6\left(cdn\frac{x-a}{b-a}\right) - \frac{1}{d^2c^4} B_6 \right.}$$

$$\boldsymbol{- \ \frac{1}{d^2} \hat{B}_6\left(dn\frac{x-a}{b-a}\right) + \frac{1}{d^2} B_6}$$

$$\boldsymbol{- \ \frac{1}{c^4} \hat{B}_6\left(cn\frac{x-a}{b-a}\right) + \frac{1}{c^4} B_6}$$

$$\boldsymbol{\left. + \ \hat{B}_6\left(n\frac{x-a}{b-a}\right) \ - B_6 \right) dx,}$$

If s equals 2, we have

$$\boldsymbol{R_{cdn} = \frac{d^4 S_{cdn} - S_{cn}}{d^4 - 1} = \frac{d^4 c^2 T_{cdn} - d^4 T_{dn} - c^2 T_{cn} + T_n}{(d^4 - 1)(c^2 - 1)}}$$

$$\boldsymbol{= \int_a^b f(x)\ dx + \ \frac{(b-a)^7 f^{(6)}(?) B_6}{6!(c^2-1)(d^4-1)n^6}\left(1 - \frac{1}{d^2}\right)\left(1 - \frac{1}{c^4}\right)}$$

$$\boldsymbol{= \int_a^b f(x)\ dx + \ \frac{(b-a)^7 f^{(6)}(?) \frac{1}{42}}{720(c^2-1)(d^4-1)n^6}\left(\frac{d^2-1}{d^2}\right)\left(\frac{c^4-1}{c^4}\right)}$$

$$\boldsymbol{= \int_a^b f(x)\ dx + \ \frac{(b-a)^7 (c^2+1)}{30240\ c^4\ d^2\ (d^2+1)\ n^6}\ f^{(6)}(?)}$$

This is the error formula for the Third Columns in Romberg Tables

A Summary of Romberg Method Remainders:

$$T_n = \int_a^b f(x)\ dx + \frac{(b-a)^3}{n^2}\frac{B_2}{2!}f^{(2)}(?)$$

$$\begin{aligned} S_{cn} &= \frac{c^2 T_{cn} - T_n}{c^2 - 1} \\ &= \int_a^b f(x)\ dx - \frac{(b-a)^5}{(c^2-1)n^4}\frac{B_4}{4!}\left(1 - \frac{1}{c^2}\right) f^{(4)}(?) \end{aligned}$$

$$R_{cdn} = \frac{d^4 S_{cdn} - S_{cn}}{d^4 - 1}$$

$$= \int_a^b f(x)\ dx + \frac{(b-a)^7}{(c^2-1)(d^4-1)n^6}\frac{B_6}{6!}\left(1 - \frac{1}{d^2}\right)\left(1 - \frac{1}{c^4}\right) \cdot f^{(6)}(?)$$

$$R_{cdn} = \frac{d^4 S_{cdn} - S_{cn}}{d^4 - 1}$$

$$= \int_a^b f(x)\ dx + \frac{(b-a)^7}{(c^2-1)(d^4-1)n^6} \frac{B_6}{6!} \left(1 - \frac{1}{d^2}\right)\left(1 - \frac{1}{c^4}\right) \cdot f^{(6)}(?)$$

$$Q_{cden} = \frac{e^6 R_{cden} - R_{cn}}{e^6 - 1}$$

$$= \int_a^b f(x)\ dx$$

$$+ \frac{(b-a)^9}{(c^2-1)(d^4-1)(e^6-1)n^8} \frac{B_8}{8!} \left(1 - \frac{1}{e^2}\right)\left(1 - \frac{1}{d^4}\right)\left(1 - \frac{1}{c^6}\right) \cdot f^{(8)}(?)$$

If $ncde$... must be equal to a fixed number of evaluation points, choose n as a larger factor and $c, d, e, ..$ as smaller factors.