

Consider two sequences:

terms $\{a_n\}$ to be added, as in a sum $\sum_{n=1}^{\infty} a_n$

and their partial sums $\left\{ s_n = \sum_{k=1}^n a_k \right\}$,

where $s_1 = a_1$,

$$s_2 = a_1 + a_2,$$

$$s_3 = a_1 + a_2 + a_3,$$

$$s_4 = a_1 + a_2 + a_3 + a_4,$$

...

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n, \text{ etc.}$$

We shall define the sum of the series $\sum_{n=1}^{\infty} a_n$ to be $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$,

if this limit exists, as a finite number.

Example: A Geometric Series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \text{ for } -1 < x < 1.$$

We shall write

$$\begin{aligned} (1-x)(1+x+\dots+x^{n-2}+x^{n-1}) &= 1+x+\dots+x^{n-2}+x^{n-1} \\ &\quad -x-x^2-\dots-x^{n-1}-x^n \\ &= 1-x^n \end{aligned}$$

as: $s_{n-1} = \sum_{k=0}^{n-1} x^k = 1+x+x^2+\dots+x^{n-1} = \frac{1-x^n}{1-x}.$

Then, we take the limit as n approaches ∞ :

$$\lim_{n \rightarrow \infty} s_n = \sum_{k=0}^{\infty} x^k = 1+x+x^2+\dots = \frac{1-0}{1-x}.$$

Example: A More General Geometric Series

If $-1 < b < 1$,

$$a + ab + ab^2 + ab^3 + ab^4 + ab^5 + ab^6 + \cdots + ab^n + \cdots$$

$$= \sum_{k=0}^{\infty} ab^k$$

$$= a \sum_{k=0}^{\infty} b^k$$

$$= a \frac{1}{1 - b}$$

$$= \frac{a}{1 - b}$$

$$= \frac{\text{first term}}{1 - \text{ratio}}$$

Another Example: A Telescoping Series

By letting n approach ∞ in the identity

$$\begin{aligned}\sum_{k=a}^n (f(k) - f(k+1)) &= \sum_{k=a}^n f(k) - \sum_{k=a}^n f(k+1) \\ &= \sum_{k=a}^n f(k) - \sum_{k=a+1}^{n+1} f(k) = f(a) - f(n+1),\end{aligned}$$

we can have

A Telescoping Series

$$\text{If } \lim_{n \rightarrow \infty} f(n) = 0, \quad \text{then } \sum_{k=a}^{\infty} (f(k) - f(k+1)) = f(a).$$

Example:
$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) = \frac{1}{1^2} = 1$$

$$\begin{aligned}
\text{Similarly, } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \lim_{b \rightarrow \infty} \sum_{n=1}^b \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \lim_{b \rightarrow \infty} \left(\sum_{n=1}^b \frac{1}{n} - \sum_{n=1}^b \frac{1}{n+1} \right) \\
&= \lim_{b \rightarrow \infty} \left(\sum_{k=1}^b \frac{1}{k} - \sum_{k=2}^{b+1} \frac{1}{k} \right) \\
&= \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b+1} \right) = 1.
\end{aligned}$$

Warning! Do not oversimplify it like this:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} \\
&= \infty - \infty = 0.
\end{aligned}$$

A Simple Theorem:

If the series $\sum_{n=1}^{\infty} a_n$ converges,
then the sequence $\{a_n\}$ approaches zero.

Proof:

Let L equal $\sum_{n=1}^{\infty} a_n$.

Then a_n equals $\sum_{k=1}^n a_k - \sum_{k=1}^{n-1} a_k$,
which approaches $L - L = 0$.

Warning!: The converse is not necessarily true!

It is possible to have $\{a_n\} \rightarrow 0$, while $\sum_{n=1}^{\infty} a_n$ diverges.

For Example, it is possible to have a sequence, like $\frac{1}{n}$, converge to zero,
 while its series : $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞ .

Consider the partial sum $\sum_{k=1}^{2^n} \frac{1}{k}$:

$$\begin{aligned}
 & \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \frac{1}{9} + \dots + \frac{1}{2^n} \\
 > & \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \frac{1}{16} + \dots + \frac{1}{2^n} \\
 = & \frac{1}{1} + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \dots + \frac{2^{n-1}}{2^n} \\
 = & \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} \\
 = & \frac{1}{1} + \frac{n}{2}, \text{ which would } \rightarrow \infty \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Alternating Series

If all a_k in a sequence are nonnegative, $a_k \geq 0$,
if they are nonincreasing,

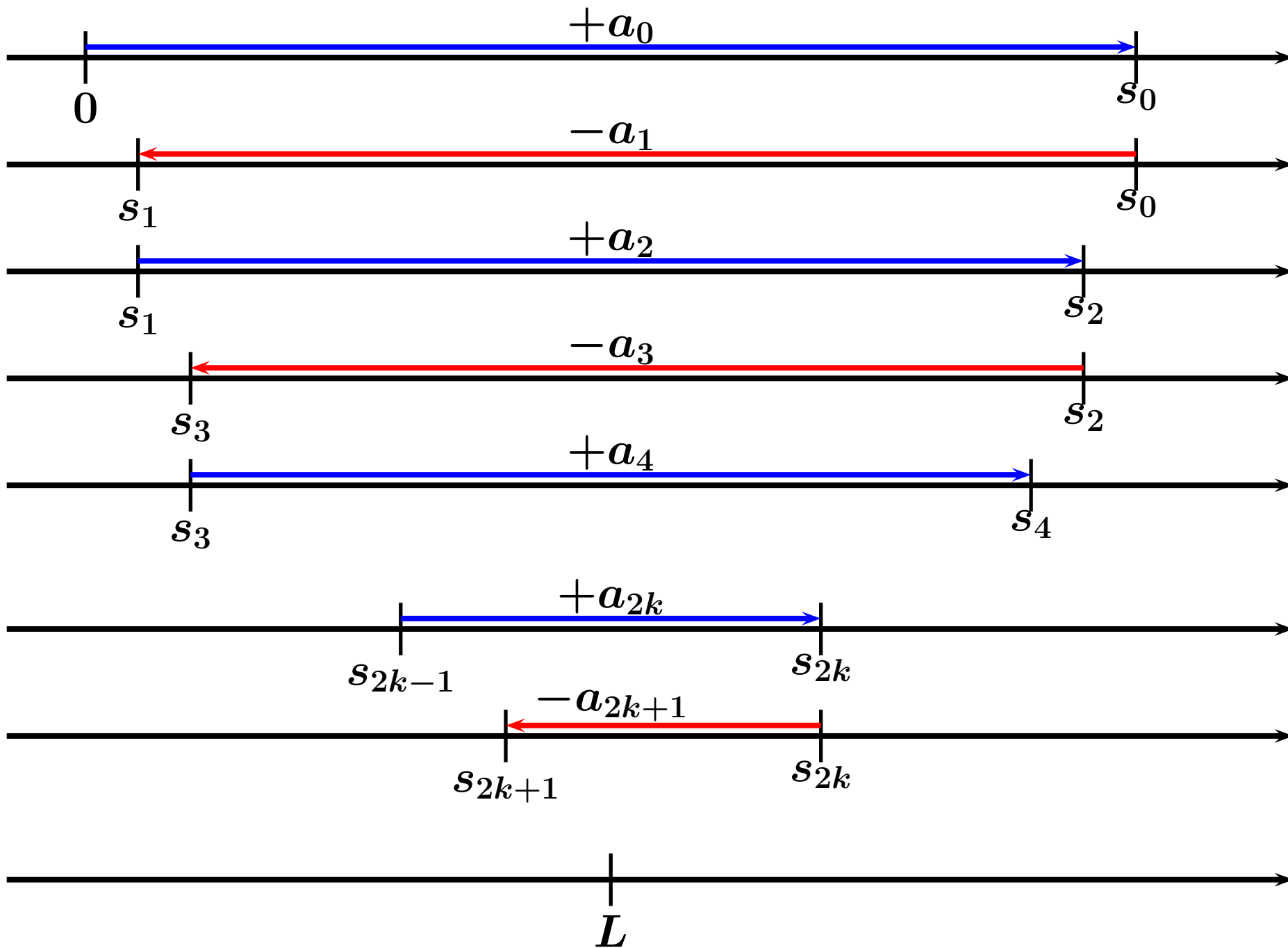
$$a_0 \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq \cdots \geq a_n \geq \cdots \cdots,$$

and if they approach zero, $\lim_{n \rightarrow \infty} a_n = 0$,

then the alternating series $\sum_{n=0}^{\infty} (-1)^n a_n$ converges:

$$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + \cdots \pm a_n \mp \cdots \cdots$$

(How fast the terms a_n approach zero makes no difference.)

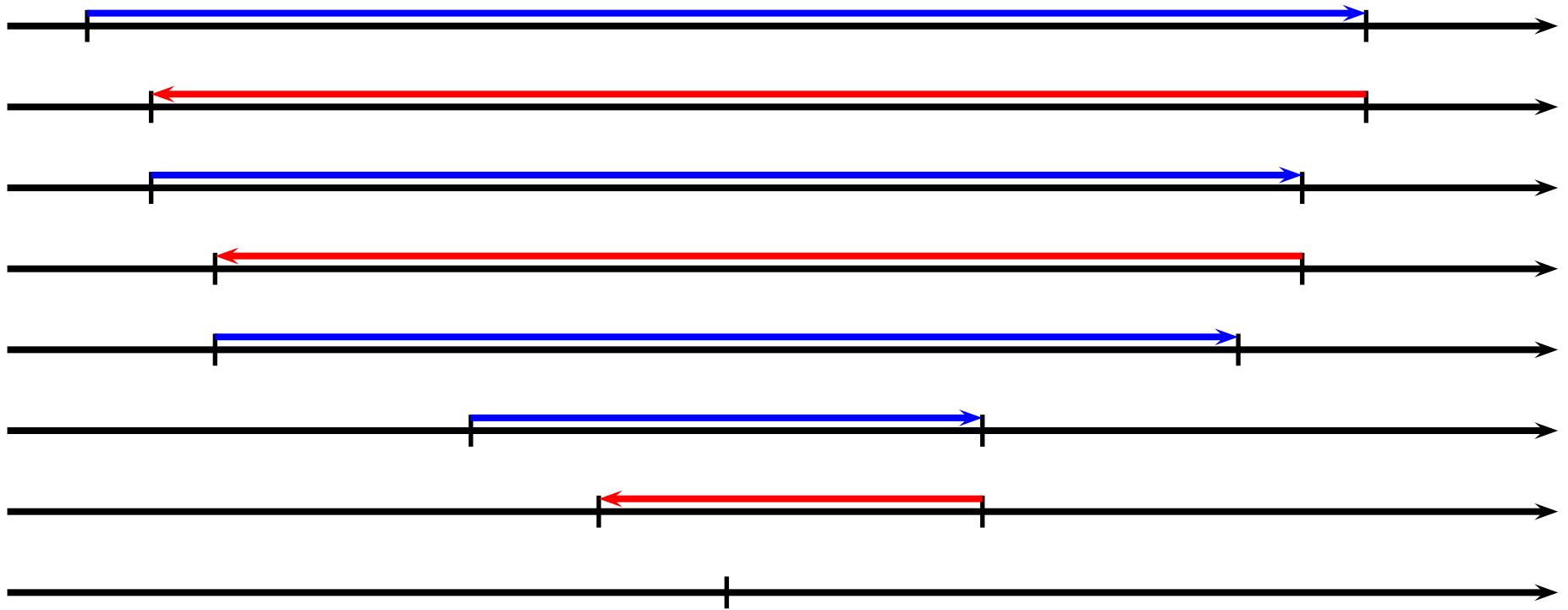


The “zigs” s_{2k} are nonincreasing with the “zags” as lower bounds.

They approach their greatest lower bound, $\lim s_{2k}$.

The “zags” s_{2k+1} are nondecreasing with “zigs” as lower bounds.

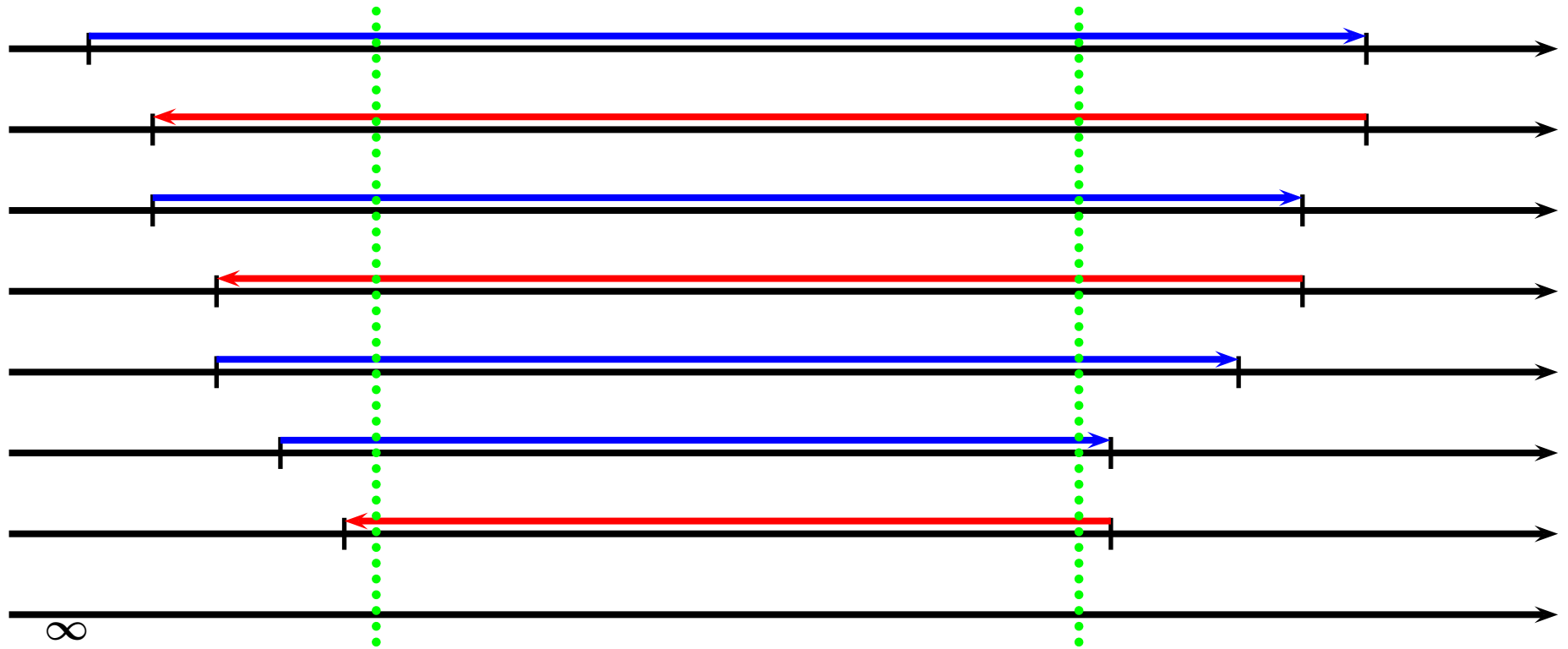
They approach their least upper bound, $\lim s_{2k+1}$.



Both of these limits are crossed by each a_n ,

which approach zero. These limits coincide as $L = \lim s_n$.

If the a_n were positive and decreasing,
but not approaching zero,
then we would have $\lim s_{2k+1} < \lim s_{2k}$
and $\sum a_n$ would not converge:

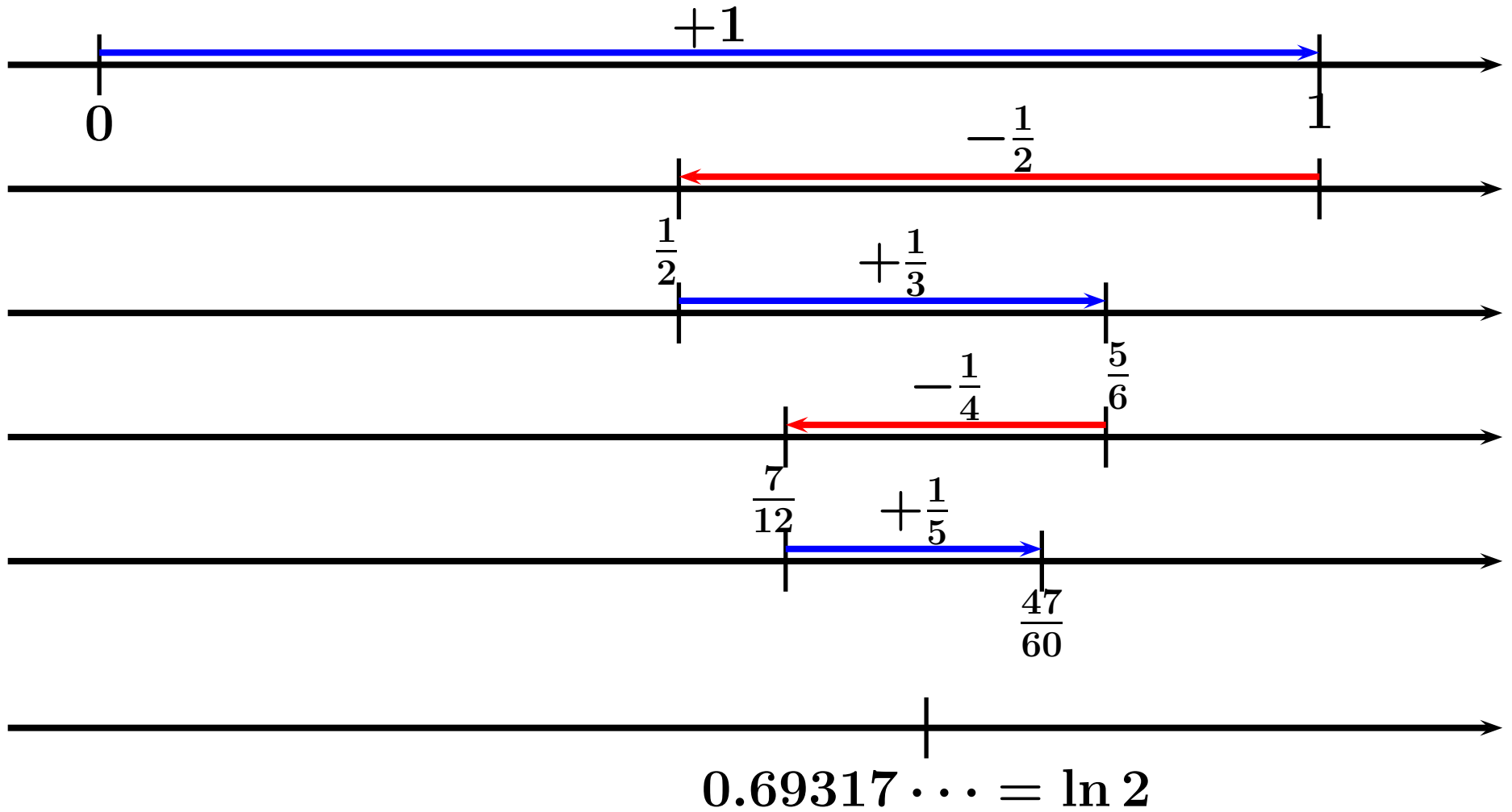


$\sum_{n=1}^{\infty} (-1)^n 2^{-\frac{1}{n}}$ would be such an example.

$$\left(2^{-\frac{1}{n}} \not\rightarrow 0 \right)$$

Example:

$$\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \text{ converges to } \ln 2.$$



Some Special Inequalities:

$$\frac{1}{n+1} \cdot 1 < \int_n^{n+1} \frac{1}{x} dx < \frac{1}{n} \cdot 1$$

$$\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$$

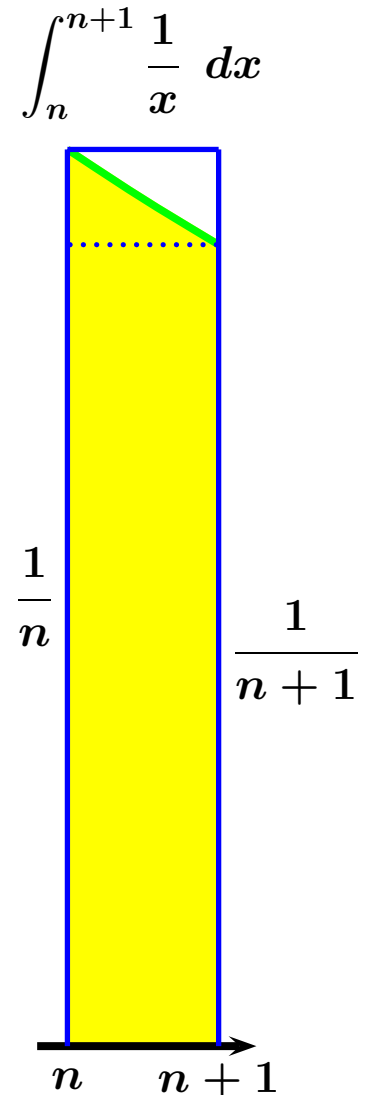
$$\frac{1}{n+1} < \ln\left(\frac{n+1}{n}\right) < \frac{1}{n}$$

A positive decreasing sequence which $\rightarrow 0$:

$$1 > \ln\left(\frac{2}{1}\right) > \frac{1}{2} > \ln\left(\frac{3}{2}\right) > \frac{1}{3} > \ln\left(\frac{4}{3}\right) > \frac{1}{4} \dots$$

An alternating series which converges:

$$1 - \ln\left(\frac{2}{1}\right) + \frac{1}{2} - \ln\left(\frac{3}{2}\right) + \frac{1}{3} - \ln\left(\frac{4}{3}\right) + \frac{1}{4} \dots$$



An alternating series which converges:

$$1 - \ln\left(\frac{2}{1}\right) + \frac{1}{2} - \ln\left(\frac{3}{2}\right) + \frac{1}{3} - \ln\left(\frac{4}{3}\right) + \frac{1}{4} - \dots$$

A limit which converges:

$$\lim_{n \rightarrow \infty} \left(1 - \ln\left(\frac{2}{1}\right) + \frac{1}{2} - \ln\left(\frac{3}{2}\right) + \frac{1}{3} - \dots - \ln\left(\frac{n}{n-1}\right) + \frac{1}{n} \right)$$

$$= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right. \\ \left. - \ln\left(\frac{2}{1}\right) - \ln\left(\frac{3}{2}\right) - \dots - \ln\left(\frac{n}{n-1}\right) \right)$$

$$= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \dots \cdot \frac{n}{n-1}\right) \right)$$

A limit which converges:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \dots \frac{n}{n-1} \right) \right) \\ &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right) \\ &= 0.5772156649015328606065120900824024310422\dots \end{aligned}$$

This is called the Euler Constant, γ .

After about 270 years,
we still haven't been able to prove
whether γ is rational (unlikely), or irrational (likely).

Here are some uses for the Euler Constant, γ .

$$\int_0^{\infty} \left(\frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x} dx = \gamma$$

$$\int_0^{\infty} \frac{1}{x} \left(\frac{1}{1 + x} - e^{-x} \right) dx = \gamma$$

$$\int_0^{\infty} e^{-x} \ln x dx = -\gamma$$

$$\int_0^{\infty} e^{-x^2} \ln x dx = -\frac{\sqrt{\pi}}{4} (\gamma + 2 \ln 2)$$

$$\frac{d}{dx} \int_{t=0}^{\infty} e^{-t} t^{x-1} dt = -\gamma, \text{ at } x = 1.$$

For a while, we shall concentrate on sums of nonnegative terms.

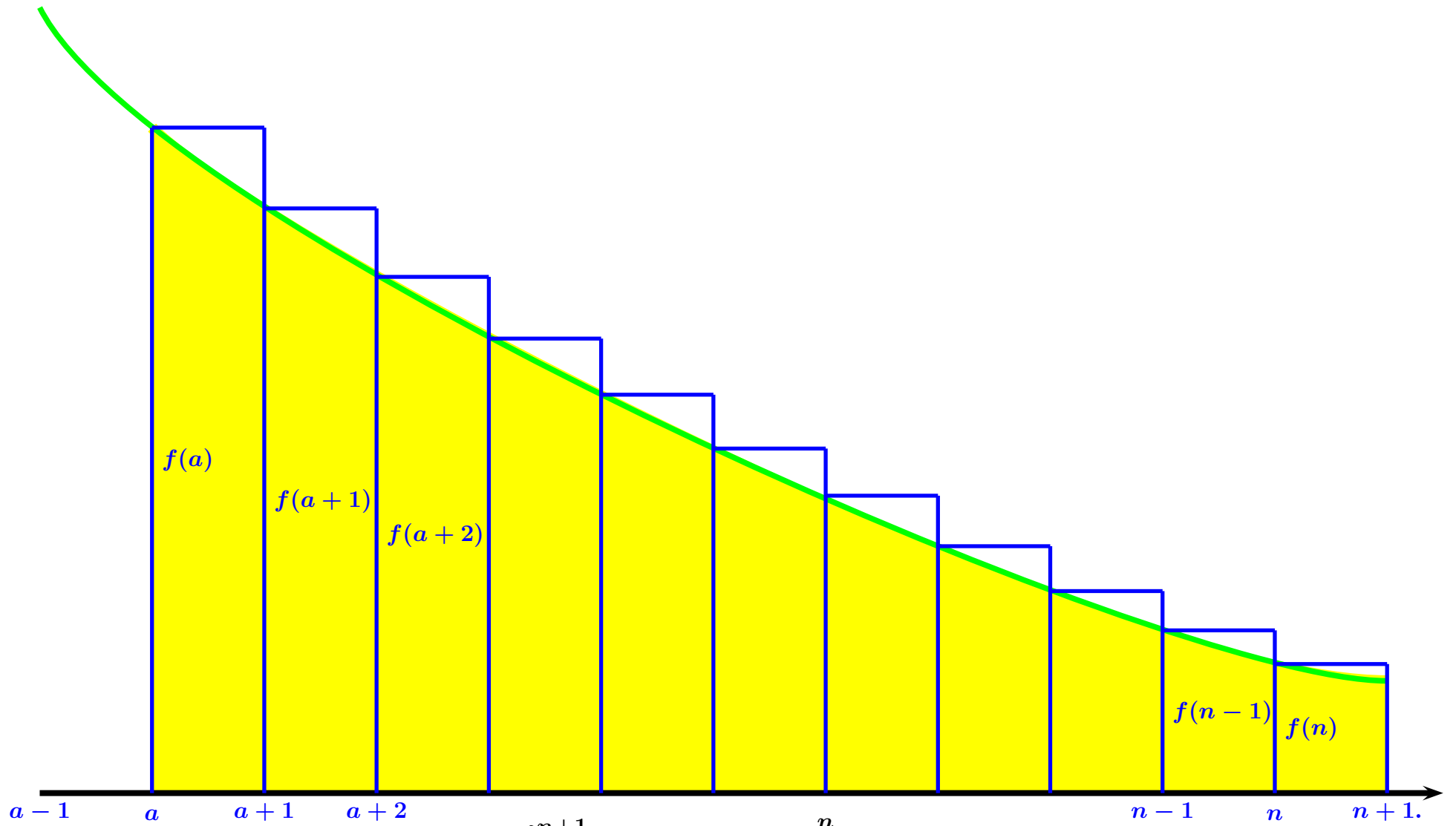
Consider now any nonnegative and nonincreasing function $f(x)$.

Recall that any nondecreasing function either approaches a finite limit or diverges to ∞ , so that

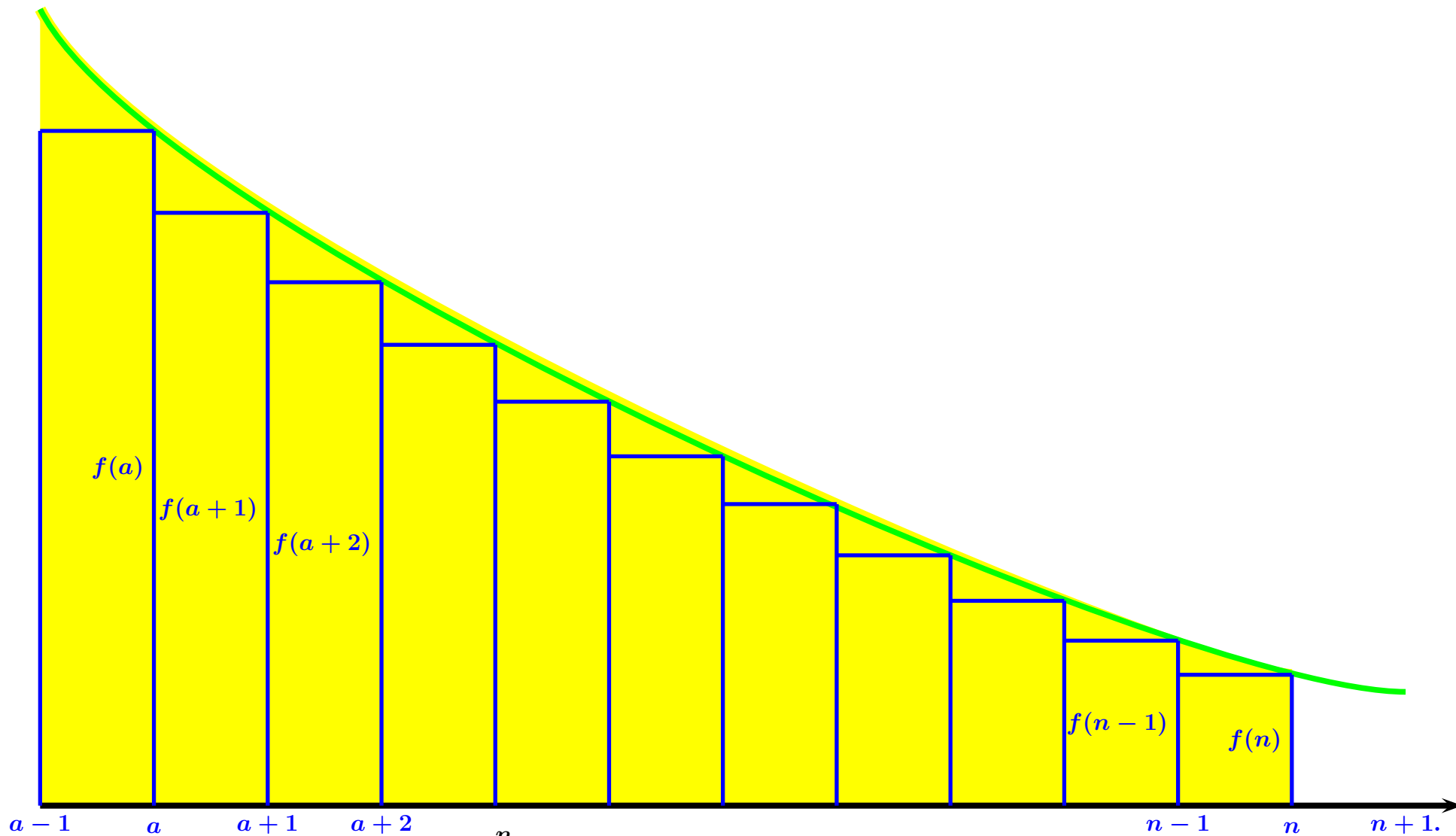
Every nonnegative function f satisfies $\int_{t=a}^{\infty} f(t) dt = L$, for some $L < \infty$,
or $\int_{t=a}^{\infty} f(t) dt$ diverges to infinity: $\int_{t=a}^{\infty} f(t) dt = \infty$.

We have a similar dichotomy for positive series:

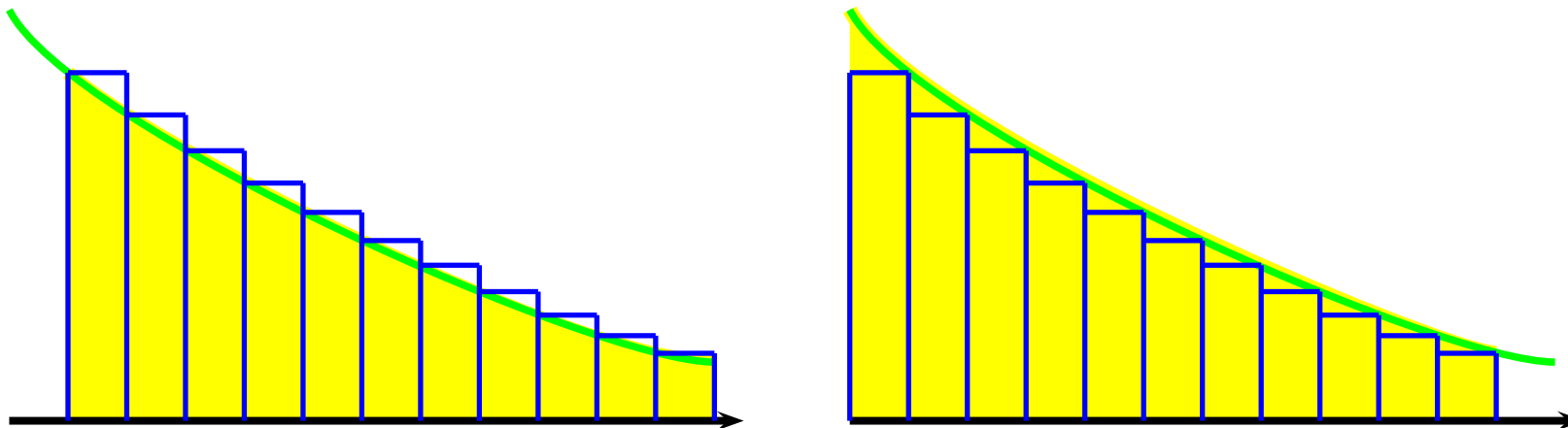
Every nonnegative function f satisfies $\sum_{k=a}^{\infty} f(k) = M$, for some $M < \infty$,
or $\sum_{k=a}^{\infty} f(k)$ diverges to infinity: $\sum_{k=a}^{\infty} f(k) = \infty$.



$$\int_{k=a}^{n+1} f(x) \, dx \leq \sum_{k=a}^n f(k)$$



$$\sum_{k=a}^n f(k) \leq \int_{k=a-1}^n f(x) dx$$



Since every nonnegative, nonincreasing function $f(x)$ satisfies

$$\int_{k=a}^{n+1} f(x) \, dx \leq \sum_{k=a}^n f(k) \leq \int_{k=a-1}^n f(x) \, dx ,$$

then $\sum_{k=a}^{\infty} f(k)$ converges iff $\int_{k=a}^{\infty} f(x) \, dx$ converges,

and $\sum_{k=a}^{\infty} f(k)$ diverges iff $\int_{k=a}^{\infty} f(x) \, dx$ diverges.

All of the following comparison tests,
which are already known for infinite integrals
of nonincreasing nonnegative functions,
apply also to infinite series of nonincreasing nonnegative terms

We will usually know more about convergence of $\int_{k=a}^{\infty} f(x) dx$
than about the convergence of $\sum_{k=a}^{\infty} f(k)$.

These tests can often help fill in
what we need to know about series convergence.

Textbooks usually call this connection the “Integral Test” for series.

The following examples follow from the Inequality Comparison Tests:

$$\text{Since } \frac{1}{x^4 + 13} \leq \frac{1}{x^4}, \quad \int_1^{\infty} \frac{1}{x^4 + 13} dx < \int_1^{\infty} \frac{1}{x^4} dx < \infty,$$

$$\int_1^{\infty} \frac{1}{x^4 + 13} dx \text{ converges, and so does } \sum_1^{\infty} \frac{1}{n^4 + 13},$$

since $\frac{1}{x^4 + 13}$ is a decreasing function.

$$\text{Since } \frac{1}{\sqrt{x^4 + 13}} \leq \frac{1}{x^2}, \quad \int_1^{\infty} \frac{1}{\sqrt{x^4 + 13}} dx < \int_1^{\infty} \frac{1}{x^2} dx < \infty,$$

$$\int_1^{\infty} \frac{1}{\sqrt{x^4 + 13}} dx \text{ converges, and so does } \sum_1^{\infty} \frac{1}{\sqrt{n^4 + 13}},$$

since $\frac{1}{\sqrt{x^4 + 13}}$ is a decreasing function.

Since $\frac{1}{x-13} \geq \frac{1}{x}$, $\int_{15}^{\infty} \frac{1}{x-13} dx \geq \int_{15}^{\infty} \frac{1}{x} dx = \infty$,

$\int_{15}^{\infty} \frac{1}{x-13} dx$ diverges, and so does $\sum_{15}^{\infty} \frac{1}{n-13}$,

since $\frac{1}{x-13}$ is a decreasing function.

Since $\frac{1}{\sqrt{x-13}} \geq \frac{1}{\sqrt{x}}$, $\int_{15}^{\infty} \frac{1}{\sqrt{x-13}} dx \geq \int_{15}^{\infty} \frac{1}{\sqrt{x}} dx = \infty$,

$\int_{15}^{\infty} \frac{1}{\sqrt{x-13}} dx$ diverges, and so does $\sum_{15}^{\infty} \frac{1}{\sqrt{n-13}}$,

since $\frac{1}{\sqrt{x-13}}$ is a decreasing function.

Since $\frac{1}{\ln x} > \frac{1}{x}$ for $x \geq 2$, $\int_2^{\infty} \frac{1}{\ln x} dx$ and $\sum_2^{\infty} \frac{1}{\ln n}$ both diverge.

Limit Comparison Test for Positive Functions:

If $0 \leq f(x)$, if $0 < g(x)$ and if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} < M < \infty$,

for ultimately decreasing, or nonincreasing f and g ,

Then $\int_{x=c}^{\infty} f(x) dx$ and $\sum_c^{\infty} f(n)$ both converge if $\int_{x=c}^{\infty} g(x) dx$ converges,

and $\int_{x=c}^{\infty} g(x) dx$ and $\sum_c^{\infty} g(n)$ both diverge if $\int_{x=c}^{\infty} f(x) dx$ diverges.

Proof: Inequality comparison test with $f(x) < (M + \epsilon)g(x)$ for large x .

Limit Comparison Test for Positive Functions:

If $0 < f(x)$, if $0 < g(x)$ and if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} > M > 0$,

for ultimately decreasing, or nonincreasing, f and g

Then $\int_{x=c}^{\infty} f(x) dx$ and $\sum_c^{\infty} f(n)$ both diverge if $\int_{x=c}^{\infty} g(x) dx$ diverges,

and $\int_{x=c}^{\infty} g(x) dx$ and $\sum_c^{\infty} g(n)$ both converge if $\int_{x=c}^{\infty} f(x) dx$ converges.

If $0 < f(x)$, $0 < g(x)$, and if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$,
then we shall say that f and g are asymptotic as $x \rightarrow \infty$
and we shall write $f(x) \sim g(x)$.

Examples: $x \sim x + 1$, since $\frac{x}{x + 2} \rightarrow 1$, $x^2 \sim (x + 1)^2$, $x^2 \sim x^2 + 1$

The Limit Comparison Tests above combine to yield another

Limit Comparison Test for Positive Functions:

If $0 < f(x)$, if $0 < g(x)$

if f and g are ultimately decreasing, or nonincreasing,

and if $f(x) \sim g(x)$,

then $\int_{x=c}^{\infty} f(x) dx$ and $\sum_c^{\infty} f(n)$ both converge if $\int_{x=c}^{\infty} g(x) dx$ converges,

and $\int_{x=c}^{\infty} f(x) dx$ and $\sum_c^{\infty} f(n)$ both diverge if $\int_{x=c}^{\infty} g(x) dx$ diverges.

Since $\frac{1}{x^4 \pm 13} \sim \frac{1}{x^4}$, and since $\int_1^\infty \frac{1}{x^4} dx < \infty$,

$\int_1^\infty \frac{1}{x^4 \pm 13} dx$ converges, and so does $\sum_1^\infty \frac{1}{n^4 \pm 13}$.

since $\frac{1}{x^4 \pm 13}$ is a decreasing function.

Since $\frac{1}{\sqrt{x^4 \pm 13}} \sim \frac{1}{x^2}$, and since $\int_1^\infty \frac{1}{x^2} dx < \infty$,

$\int_1^\infty \frac{1}{\sqrt{x^4 \pm 13}} dx$ converges, and so does $\sum_1^\infty \frac{1}{\sqrt{n^4 \pm 13}}$,

since $\frac{1}{\sqrt{x^4 \pm 13}}$ is a decreasing function.

Since $\frac{1}{x \pm 13} \sim \frac{1}{x}$, and since $\int_{15}^{\infty} \frac{1}{x} dx = \infty$,
 $\int_{15}^{\infty} \frac{1}{x \pm 13} dx$ diverges, and so does $\sum_{15}^{\infty} \frac{1}{n \pm 13}$.

Since $\frac{1}{\sqrt{x \pm 13}} \sim \frac{1}{\sqrt{x}}$, and since $\int_{15}^{\infty} \frac{1}{\sqrt{x}} dx = \infty$,
 $\int_{15}^{\infty} \frac{1}{\sqrt{x \pm 13}} dx$ diverges, and so does $\sum_{15}^{\infty} \frac{1}{\sqrt{n \pm 13}}$.

Since $\frac{\frac{1}{\ln x}}{\frac{1}{x}} = \frac{x}{\ln x} \rightarrow \infty$, $\int_2^{\infty} \frac{1}{\ln x} dx$ and $\sum_2^{\infty} \frac{1}{\ln n}$ both diverge.

To tell whether $\int_1^{\infty} \frac{x}{3^x} dx$ converges,

Compare $\frac{x}{3^x}$ with $\frac{1}{2^x}$, which lies between $\frac{1}{3^x}$ and $\frac{1}{1^x} = 1$.

Since $\lim_{x \rightarrow \infty} \frac{\frac{x}{3^x}}{\frac{1}{2^x}} = \lim_{x \rightarrow \infty} \frac{x}{\left(\frac{3}{2}\right)^x} = 0$, and since $\int_1^{\infty} \frac{1}{2^x} dx < \infty$,

then $\int_1^{\infty} \frac{x}{3^x} dx$ and $\sum_1^{\infty} \frac{n}{3^n}$ both converge.

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2^{-n}

3^{-n} ← $\left\{ \begin{array}{l} \frac{n}{3^n} \\ \frac{1}{3^n} \\ \frac{1}{3^{n^2}} \end{array} \right.$

5^{-n}

$a_n \rightarrow 0$
 $\sum a_n < \infty$ (Converges)

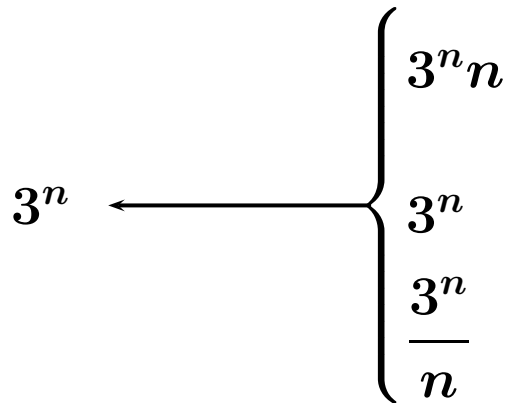
To tell whether $\int_1^{\infty} \frac{3^x}{x} dx$ converges,

compare $\frac{3^x}{x}$ with 2^x , which lies between 3^x and $1^x = 1$.

Since $\lim_{x \rightarrow \infty} \frac{3^x}{2^x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{3}{2}\right)^x}{x} = \infty$, and since $\int_1^{\infty} 2^x dx = \infty$,

then $\int_1^{\infty} \frac{3^x}{x} dx$ and $\sum_{n=1}^{\infty} \frac{3^n}{n}$ both diverge.

5^n



$$a_n \not\rightarrow 0$$

$$\sum a_n = \infty \text{ (Diverges)}$$

2^n

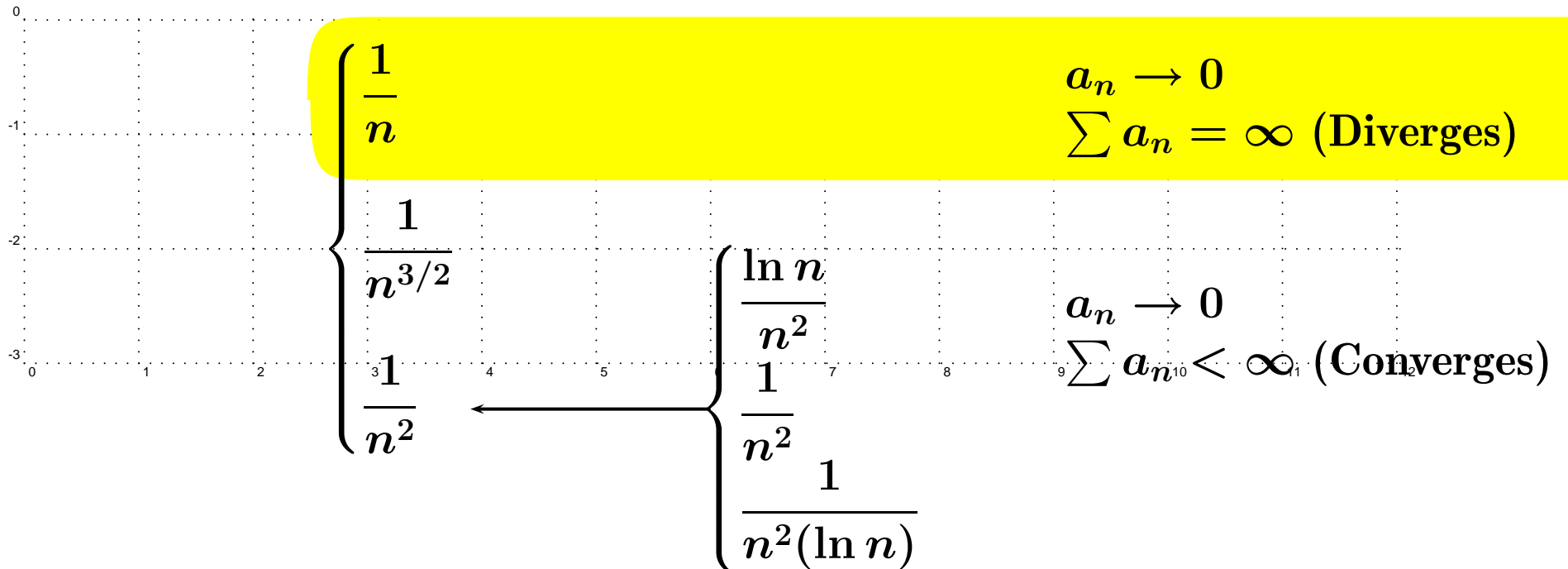
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To tell whether $\int_1^{\infty} \frac{\ln x}{x^2} dx$ converges,

compare $\frac{\ln x}{x^2}$ with $\frac{1}{x^{\frac{3}{2}}}$, which lies between $\frac{1}{x^2}$ and $\frac{1}{x^1}$.

Since $\lim_{x \rightarrow \infty} \frac{\frac{\ln x}{x^2}}{\frac{1}{x^{\frac{3}{2}}}} = \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = 0$, and since $\int_{15}^{\infty} \frac{1}{x^{\frac{3}{2}}} dx < \infty$,

then $\int_{15}^{\infty} \frac{\ln x}{x^2} dx$ and $\sum_{15}^{\infty} \frac{\ln n}{n^2}$ both converge.



To tell whether $\int_1^{\infty} \frac{1}{\sqrt{x} \ln x} dx$ converges,

compare $\frac{1}{\sqrt{x} \ln x}$ with $\frac{1}{x^{\frac{2}{3}}}$, which lies between $\frac{1}{x^{\frac{1}{2}}}$ and $\frac{1}{x^1}$.

Since $\lim_{x \rightarrow \infty} \frac{\frac{1}{\sqrt{x} \ln x}}{\frac{1}{x^{\frac{2}{3}}}} = \lim_{x \rightarrow \infty} \frac{x^{\frac{1}{6}}}{\ln x} = \infty$, and since $\int_{15}^{\infty} \frac{1}{x^{\frac{2}{3}}} dx = \infty$,

then $\int_{15}^{\infty} \frac{1}{\sqrt{x} \ln x} dx$ and $\sum_{15}^{\infty} \frac{1}{\sqrt{n} \ln n}$ both diverge.

$\left\{ \begin{array}{l} \frac{1}{\sqrt{n}} \\ \frac{1}{n^{2/3}} \\ \frac{1}{n} \end{array} \right\} \left\{ \begin{array}{l} \frac{\ln n}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n} \ln n} \end{array} \right.$

$f(x) \rightarrow 0$
 $\int f(x) dx = \infty$ (Diverges)

A Master Benchmark Example:

The convergence or divergence of

$$\int_{x=e^{e^e}}^{\infty} \frac{C^x}{x^a (\ln x)^b (\ln(\ln x))^c \left(\ln(\ln(\ln x)) \right)^d \left(\ln(\ln(\ln(\ln x))) \right)^f} dx$$

is determined by the first of C, a, b, c, d, f to differ from 1.

Otherwise, it diverges.

Another Master Benchmark Example:

The convergence or divergence of

$$\sum_{n=\lceil e^{e^e} \rceil}^{\infty} \frac{C^n}{n^a (\ln n)^b (\ln(\ln n))^c \left(\ln(\ln(\ln n)) \right)^d \left(\ln(\ln(\ln(\ln n))) \right)^f}$$

is determined by the first of C, a, b, c, d, f to differ from 1.

Otherwise, it diverges.