Recall:

If a series $\sum_{n=1}^{\infty} a_n$ **converges absolutely,** $\sum_{n=1}^{\infty} |a_n| < \infty$,
then $\sum_{n=1}^{\infty} a_n$ **also converges in the original definition.**

If a series $\sum_{n=1}^{\infty} a_n$ **converges, then** $a_n \longrightarrow 0$**, as** $n \longrightarrow \infty$.

If sequence of terms converges to zero, $a_n \longrightarrow 0$**, as** $n \longrightarrow \infty$.
then the their absolute values also converges to zero,
$|a_n| \longrightarrow 0$**, as** $n \longrightarrow \infty$,
and they are bounded by some positive M: $|a_n| \leq M$.

A Power Series is a series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$ **.**

Its sum can define a function of x**, wherever it converges, usually on an interval centered at the number** c**.**

$\sum_{n=0}^{\infty} a_n(x-c)^n$ **is often called a Taylor Series.**

If c **equals 0,** $\sum_{n=0}^{\infty} a_n x^n$ **is often called a Maclaurin Series.**

Brook Taylor and Colin Maclaurin were younger friends of Isaac Newton, the actual inventor of power series.

For simplicity, much of the following will be written in terms of Maclaurin series but will apply also WOLG to Taylor Series.

Theorem:

If, for some x,

a power series $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely, converges,

or even has its sequence of terms bounded, $|a_n x^n| \le M$,

then at any smaller x_1, $\quad |x_1| < |x|$,

the power series $\sum_{n=1}^{\infty} a_n x_1^n$ will converge absolutely, converge,

and its terms $a_n x_1^n$ will also approach zero and be bounded.

Proof: $$\sum_{n=1}^{\infty} |a_n x_1^n| = \sum_{n=1}^{\infty} |a_n x^n| \left|\frac{x_1}{x}\right|^n \le \sum_{n=1}^{\infty} M \left|\frac{x_1}{x}\right|^n < \infty,$$

if we use the inequality comparison test for positive series.

Theorem:

If. for some x_1,

a power series $\sum_{n=1}^{\infty} a_n x_1^n$ does not converge absolutely,

or if it diverges,

or if its terms $a_n x_1^n$ fail to approach zero or are unbounded,

then at any larger x, $\quad |x| > |x_1|$,

the power series $\sum_{n=1}^{\infty} a_n x^n$ will not converge absolutely,

nor will it converge in the original sense,

nor will its sequence of terms, $a_n x^n$, be bounded by any M.

Proof: This is just the contrapositive to the last Theorem.

If x equals zero, a series $\sum_{n=0}^{\infty} a_n x^n = a_0 + 0 + 0 + \ldots$ converges trivially. $\sum_{n=0}^{\infty} n! x^n$ converges only at $x = 0$.

For a series like $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, convergence occurs for all x.

Many series converge for some x's and diverge for other x's. The theorems above show that the absolute values of x where a given Maclauren series converges are all $\leq$ than the absolute values of x's where divergence takes place. As a nonempty bounded set, this set of absolute values of convergent x's has a least upper bound, called the Radius of Convergence R of the series. If convergence takes place for all x, we say that $R = \infty$.

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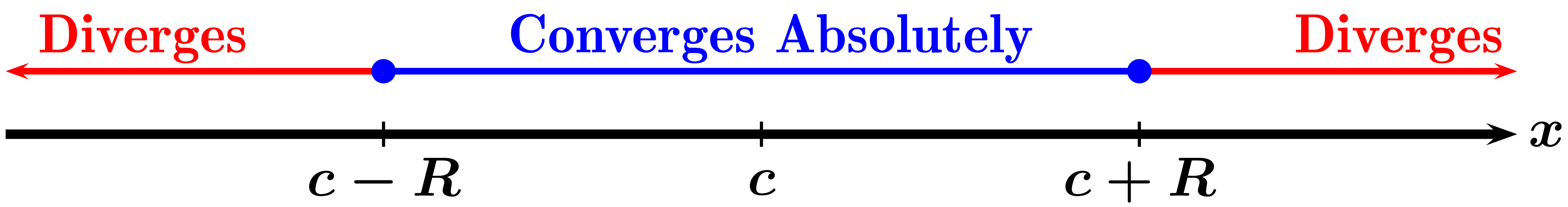
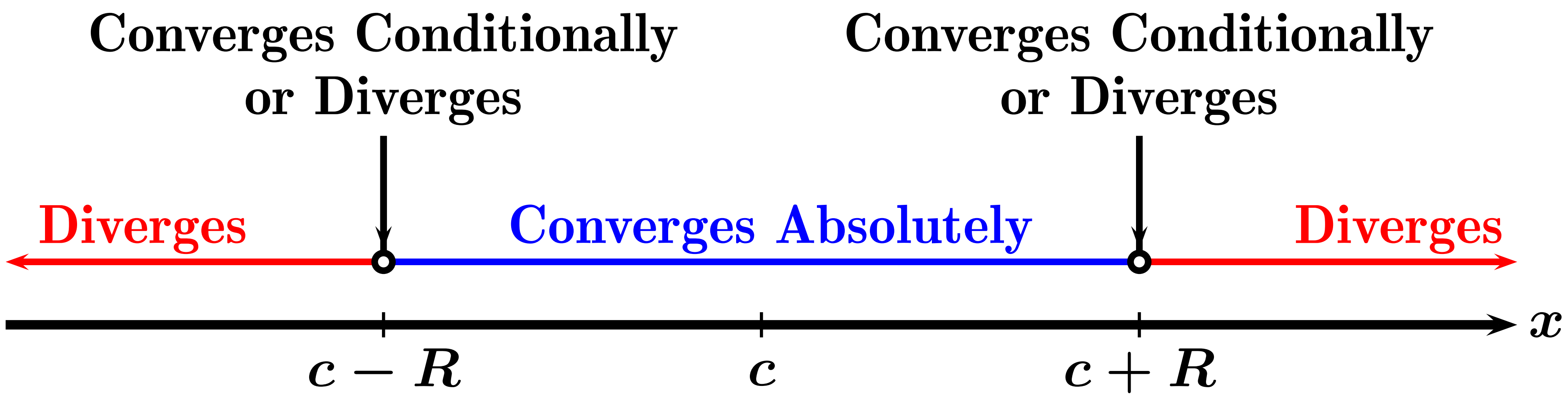
called the Radius of Convergence R of the series.
If convergence takes place for all x, we say that $R = \infty$.

As an example,
$\sum_{k=0}^{\infty} x^n$ **converges for $|x| < 1$ and diverges for $|x| \geq 1$.**
Its Radius of Convergence equals 1.

(Anything can happen
when $|x|$ equals the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$,
absolute convergence, conditional convergence, divergence.)

$$\sum_{n=1}^{\infty} a_n(x-c)^n \text{ often behaves in one of these ways:}$$





The Radius of Convergence R of $\sum_{k=0}^{n} a_n x^n$

equals the least upper bound of $|x|$ where $\sum_{k=0}^{n} a_n x^n$ converges,

equals the *lub* of $|x|$ where $\sum_{k=0}^{n} \left|a_n x^n\right|$ converges,

equals the *lub* of $|x|$ where $\left|a_n x^n\right|$ approaches 0,

equals the *lub* of $|x|$ where $\left\{\left|a_n x^n\right| \leq 1 \text{ for } n \geq \text{some } k\right\}$,

equals the *lub* of $|x|$ where $\left\{\sqrt[n]{\left|a_n x^n\right|} \leq 1 \text{ for } n \geq \text{some } k\right\}$,

The Radius of Convergence R of $\sum_{k=0}^{n} a_n x^n$

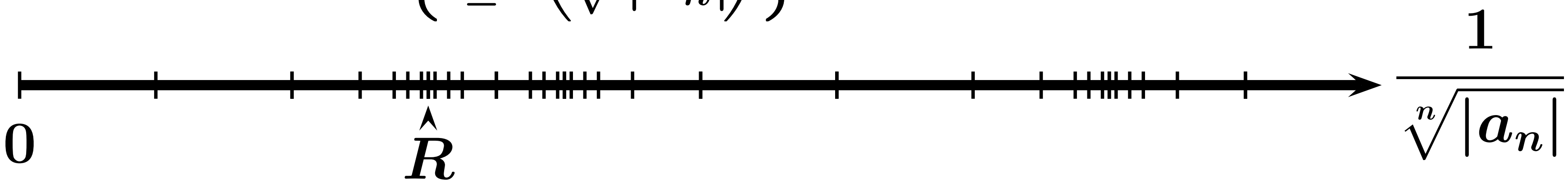
$$\text{equals the } lub \text{ of } |x| \text{ where } \left\{ \sqrt[n]{|a_n x^n|} \le 1 \text{ for } n \ge \text{some } k \right\},$$

$$\text{equals the } lub \text{ of } |x| \text{ where } \left\{ |x| \le \frac{1}{\sqrt[n]{|a_n|}} \text{ for } n \ge \text{some } k \right\},$$

$$\text{equals the } lub \text{ of } |x| \text{ where } \left\{ |x| \le \underset{n \ge k}{glb} \left(\frac{1}{\sqrt[n]{|a_n|}} \right) \text{ for some } k \right\},$$

$$\text{equals the } lub \text{ of } |x| \text{ where } |x| < \underset{k \ge 1}{lub} \left\{ \underset{n \ge k}{glb} \left(\frac{1}{\sqrt[n]{|a_n|}} \right) \right\},$$

$$\text{equals the } \underset{k \ge 1}{lub} \left\{ \underset{n \ge k}{glb} \left(\frac{1}{\sqrt[n]{|a_n|}} \right) \right\}. \qquad \text{(Cauchy-Hadamard)}$$



Many series are simple enough to use the following:

If $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ **exists, whether it is 0, positive or infinite,**

its "reciprocal" R **is the radius of convergence of** $\sum_{k=0}^{n} a_n x^n$.

Root Test: $\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|}|x| \leq 1$

$$\text{iff } |x| \leq \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$$

Examples:

$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n$ **has radius of convergence** $\frac{1}{2}$.

$\sum_{n=0}^{\infty} (\ln n)^2 \frac{x^n}{2^n} = \sum_{n=0}^{\infty} (\ln n)^2 \left(\frac{x}{2}\right)^n$ **has radius of convergence 2.**

Examples:

$$\sum_{k=0}^{\infty} 5^{2k}x^{2k} \text{ can be considered as } \sum_{k=0}^{\infty}(25x^2)^k = 1 + 25x^2 + \ldots$$

which converges for $|x^2| < \frac{1}{25}$, or for $|x| < \frac{1}{5}$.

$$\sum_{k=0}^{\infty} 7^{2k+1}x^{2k+1} \text{ can be considered as } \quad 7x\sum_{k=0}^{\infty}(49x^2)^k$$

which converges for $|x^2| < \frac{1}{42}$, or for $|x| < \frac{1}{7} = 7x + 7^3x^3 + \ldots$.

Added together, they become $\displaystyle\sum_{n=0}^{\infty} a_n x^n = 1 + 7x + 5^2x^2 \ldots,$

where $a_n = 5^n$, $\frac{1}{\sqrt[n]{|a_n|}} = \frac{1}{5}$ for even n,

$a_n = 7^n$, $\frac{1}{\sqrt[n]{|a_n|}} = \frac{1}{7}$ for odd n, and $\displaystyle \underset{L\geq 1}{lub}\left\{\underset{n\geq L}{glb}\left(\frac{1}{\sqrt[n]{|a_n|}}\right)\right\} = \frac{1}{7}.$

Usual Procedure for determining the x's

where the Taylor Series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges:

Find the Convergence Radius R as $\dfrac{1}{\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}}$ or as $\dfrac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}$.

(The series converges absolutely for $c - R < x < c + R$.)

Check for absolute convergence at both endpoints $x = c \pm R$

by using comparison tests for positive series on $\sum_{k=0}^{\infty} |a_n| R^n$.

If $\sum_{n=0}^{\infty} |a_n| R^n = \infty$ check at $x = c \pm R$ for (conditional)

convergence on $\sum_{n=0}^{\infty} a_n(\pm R)^n$. (Alternating series)

Example: Consider the series for $\ln x$, $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-1)^k}{k}$:

Find its Radius of Convergence:

$$\sqrt[k]{\left|(-1)^{k+1}\frac{(x-1)^k}{k}\right|} = \frac{|x-1|}{\sqrt[k]{k}} \quad \rightarrow \quad |x-1| < 1.\ R = \tfrac{1}{1} = 1$$

Check for Absolute Convergence at $|x-1| = 1$:

$$\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{(x-1)^k}{k}\right| = \sum_{k=1}^{\infty}\frac{|x-1|^k}{k} = \sum_{k=1}^{\infty}\frac{1}{k} = \infty.$$ **Nope.**

Check for Conditional Convergence at $(x-1) = 1, x = 2$:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1^k}{k}$$ **Yes, it alternates there.**

Check for Conditional Convergence at $(x-1) = -1, x = 0$:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(-1)^k}{k} = -\sum_{k=1}^{\infty}\frac{1}{k}$$ **Nope. It diverges there.**

Example: Consider the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-1)^k}{k^2}$**:**

Find its Radius of Convergence:

$$\sqrt[k]{\left|(-1)^{k+1}\frac{(x-1)^k}{k^2}\right|} = \frac{|x-1|}{\sqrt[k]{k^2}} \rightarrow |x-1| < 1.$$

Check for Absolute Convergence at $|x-1| = 1$ **:**

$$\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{(x-1)^k}{k^2}\right| = \sum_{k=1}^{\infty}\frac{|x-1|^k}{k^2} = \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty. \text{ Yup.}$$

This series converges absolutely for $-1 \leq x \leq 1$**.**

Example: Consider the series for e^{-x}, $\displaystyle\sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$**:**

Find its Radius of Convergence:

$$\sqrt[k]{\left|\frac{(-x)^k}{k!}\right|} = \frac{|x|}{\sqrt[k]{k!}} \to 0 < 1, \text{ for any } x.\ R = \infty.$$

No Endpoints to worry about.

Example: Consider the series for $\sin x$, $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$**:**

Find its Radius of Convergence:

$$\sqrt[2k+1]{\left|(-1)^k \frac{x^{2k+1}}{(2k+1)!}\right|} = \frac{|x|}{\sqrt[2k+1]{(2k+1)!}} \to 0,$$

for any x. $R = \infty$.

Example:

The series for $(1+x)^a : 1 + \sum_{k=1}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!} x^k$:

If a **is a nonnegative integer,**

this is a polynomial of $a+1$ **terms, and** R **is infinite.**

Otherwise, $\left| \frac{\frac{a(a-1)\cdots(a-k)}{(k+1)!} x^{k+1}}{\frac{a(a-1)\cdots(a-k+1)}{k!} x^k} \right| = \frac{|a-k||x|}{(k+1)} \to |x|, \ \ R = 1.$

At $|x| = 1$**,we have**

$$\left| \frac{a(a-1)\cdots(a-k+1)}{k!} x^k \right| = \frac{|a||a-1|\cdots|k-a-1|}{k!}$$

$$\sim C \frac{\sqrt{2\pi(k-a-1)}(k-a-1)^{k-a-1} e^{-(k-a-1)}}{\sqrt{2\pi k}\ k^k\ e^{-k}} \sim \frac{C}{k^{a+1}},$$

with absolute convergence if $a > 0$,

and conditional convergence only at $x = 1$ **when** $-1 < a < 0$.