A Partial Fraction Series for the Cotangent Function and An Infinite Product for the Sine Function

Consider the convergent series

$$f(x) = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2\pi^2} = \frac{1}{x} + \sum_{k=1}^{\infty} \left(\frac{1}{x - k\pi} + \frac{1}{x + k\pi} \right),$$

which is defined for all real x except multiples of π.

Its derivative equals

$$f'(x) = -\frac{1}{x^2} + \sum_{k=1}^{\infty} \left(-\frac{1}{(x - k\pi)^2} - \frac{1}{(x + k\pi)^2} \right),$$

$$f'(x) = -\sum_{k=-\infty}^{\infty} \frac{1}{(x - k\pi)^2}.$$

To justify the series differentiation above, check the pattern

$$g(u) \quad = \quad g(x) \quad + \quad g'(x)(u-x) + o(u-x):$$

$$\frac{1}{u \pm k\pi} \quad = \quad \frac{1}{x \pm k\pi} \quad - \quad \frac{(u-x)}{(x \pm k\pi)^2} \quad + \quad \frac{(u-x)^2}{(u \pm k\pi)(x \pm k\pi)^2}$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{u - k\pi} + \frac{1}{u + k\pi} \right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{x - k\pi} + \frac{1}{x + k\pi} \right)$$

$$+ \sum_{k=1}^{\infty} \left(-\frac{1}{(x - k\pi)^2} - \frac{1}{(x + k\pi)^2} \right)(u-x)$$

$$+ \sum_{k=1}^{\infty} \left(\frac{1}{(u - k\pi)(x - k\pi)^2} + \frac{1}{(u + k\pi)(x + k\pi)^2} \right)(u-x)^2$$

For $f(x) = \frac{1}{x} \; + \; \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2\pi^2}$ **we also need to evaluate**

$$f^2(x) = \frac{1}{x^2} \; + \; 2\frac{1}{x}\sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2\pi^2}$$
$$+ \; \sum_{K=1}^{\infty}\left(\frac{1}{x - K\pi} + \frac{1}{x + K\pi}\right)\sum_{L=1}^{\infty}\left(\frac{1}{x - L\pi} + \frac{1}{x + L\pi}\right),$$

$$f^2(x) = \frac{1}{x^2} \; + \; \sum_{k=1}^{\infty} \frac{4}{x^2 - k^2\pi^2}$$
$$+ \; \sum_{K=1}^{\infty}\left(\sum_{L=1}^{\infty}\left(\frac{1}{x - K\pi}\;\frac{1}{x - L\pi} + \frac{1}{x - K\pi}\;\frac{1}{x + L\pi}\right.\right.$$
$$\left.\left. + \; \frac{1}{x + K\pi}\;\frac{1}{x - L\pi} + \frac{1}{x + K\pi}\;\frac{1}{x + L\pi}\right)\right),$$

$$f^2(x) - \frac{1}{x^2} \; - \; \sum_{k=1}^{\infty} \frac{4}{x^2 - k^2\pi^2}$$

$$= \sum_{K=1}^{\infty} \sum_{L=1}^{\infty} \Bigg(\frac{1}{x - K\pi} \; \frac{1}{x - L\pi} + \frac{1}{x - K\pi} \; \frac{1}{x + L\pi} + \frac{1}{x + K\pi} \; \frac{1}{x - L\pi} + \frac{1}{x + K\pi} \; \frac{1}{x + L\pi} \Bigg)$$

(Now, separate out the terms where $K = L$.)

$$= \sum_{k=1}^{\infty} \left(\frac{1}{(x - k\pi)^2} + \frac{2}{(x - k\pi)(x + k\pi)} + \frac{1}{(x + k\pi)^2} \right) + \sum_{\substack{K=1 \\ K \neq L}}^{\infty} \sum_{L=1}^{\infty} \Bigg(\frac{1}{x - K\pi} \; \frac{1}{x - L\pi} + \frac{1}{x - K\pi} \; \frac{1}{x + L\pi} + \frac{1}{x + K\pi} \; \frac{1}{x - L\pi} + \frac{1}{x + K\pi} \; \frac{1}{x + L\pi} \Bigg)$$

$$f^2(x) - \sum_{k=-\infty}^{\infty} \frac{1}{(x-k\pi)^2} - \sum_{k=1}^{\infty} \frac{4+2}{x^2-k^2\pi^2}$$

(Now, separate the rest of the terms into partial fractions.)

$$= \sum_{\substack{K=1 \\ K\neq L}}^{\infty} \sum_{L=1}^{\infty} \left(\frac{1}{(K-L)\pi(x-K\pi)} + \frac{1}{(L-K)\pi(x-L\pi)} \right.$$

$$+ \frac{1}{(K+L)\pi(x-K\pi)} + \frac{1}{(-L-K)\pi(x+L\pi)}$$

$$+ \frac{1}{(-K-L)\pi(x+K\pi)} + \frac{1}{(L+K)\pi(x-L\pi)}$$

$$\left. + \frac{1}{(-K+L)\pi(x+K\pi)} + \frac{1}{(-L+K)\pi(x+L\pi)} \right)$$

$$f^2(x) - f'(x) \; - \; \sum_{k=1}^{\infty} \frac{6}{x^2 - k^2\pi^2}$$

(Now, collect similar terms.)

$$\begin{aligned} = \; & \frac{1}{\pi} \sum_{\substack{K=1 \\ K \neq L}}^{\infty} \sum_{L=1}^{\infty} \Bigg(\left(\frac{1}{(K-L)} + \frac{1}{(K+L)} \right) \frac{1}{(x - K\pi)} \\ & + \left(\frac{1}{(-K+L)} + \frac{1}{(-K-L)} \right) \frac{1}{(x + K\pi)} \\ & + \left(\frac{1}{(L-K)} + \frac{1}{(L+K)} \right) \frac{1}{(x - L\pi)} \\ & + \left(\frac{1}{(K-L)} + \frac{1}{(-K-L)} \right) \frac{1}{(x + L\pi)} \Bigg) \end{aligned}$$

$$f^2(x) - f'(x) \quad - \quad \sum_{k=1}^{\infty} \frac{6}{x^2 - k^2\pi^2}$$

$$= \frac{1}{\pi} \sum_{\substack{K=1 \\ K \neq L}}^{\infty} \sum_{L=1}^{\infty} \Bigg(\left(\frac{1}{K-L} + \frac{1}{K+L} \right) \left(\frac{1}{x - K\pi} - \frac{1}{x + K\pi} \right) + \left(\frac{1}{L-K} + \frac{1}{L+K} \right) \left(\frac{1}{x - L\pi} - \frac{1}{x + L\pi} \right) \Bigg)$$

= (using symmetry between appearence of K's and L's)

$$\lim_{M \to \infty} \frac{2}{\pi} \sum_{K=1}^{M} \sum_{\substack{L=1 \\ L \neq K}}^{M} \left(\frac{1}{K-L} + \frac{1}{K+L} \right) \left(\frac{2K\pi}{(x - K\pi)(x + K\pi)} \right)$$

$$= \lim_{M \to \infty} \sum_{K=1}^{M} \left(\sum_{\substack{L=1 \\ L \neq K}}^{M} \left(\frac{1}{K-L} + \frac{1}{K+L} \right) \left(\frac{4K}{x^2 - K^2\pi^2} \right) \right)$$

Note specifically that the sum $\displaystyle\sum_{L=1,L\neq K}^{M}\left(\frac{1}{K-L}+\frac{1}{K+L}\right)$ **equals**

$$\sum_{\substack{L=1\\L\neq K}}^{M}\frac{1}{L+K}-\sum_{\substack{L=1\\L\neq K}}^{M}\frac{1}{L-K}=\sum_{\substack{L=K+1\\L\neq 2K}}^{M+K}\frac{1}{L}-\sum_{\substack{L=1-K\\L\neq 0}}^{M-K}\frac{1}{L}$$

$$=\sum_{L=K+1}^{2K-1}\frac{1}{L}+\sum_{L=2K+1}^{M+K}\frac{1}{L}-\sum_{L=1-K}^{-1}\frac{1}{L}-\sum_{L=1}^{M-K}\frac{1}{L}$$

$$=\sum_{L=K+1}^{2K-1}\frac{1}{L}+\sum_{L=2K+1}^{M+K}\frac{1}{L}+\sum_{L=1}^{K-1}\frac{1}{L}-\sum_{L=1}^{M-K}\frac{1}{L}\Big(L\leftrightarrows -L \text{ in the third } \sum\Big)$$

$$=\sum_{L=K+1}^{2K}\frac{1}{L}+\sum_{L=2K+1}^{M+K}\frac{1}{L}+\sum_{L=1}^{K}\frac{1}{L}-\sum_{L=1}^{M-K}\frac{1}{L}-\frac{1}{2K}-\frac{1}{K}$$

$$=\sum_{L=1}^{M+K}\frac{1}{L}-\sum_{L=1}^{M-K}\frac{1}{L}-\frac{3}{2K}=\sum_{L=M-K+1}^{M+K}\frac{1}{L}-\frac{3}{2K}.$$

$$\boldsymbol{f^2(x) - f'(x) \; - \; \sum_{k=1}^{\infty} \frac{6}{x^2 - k^2\pi^2}}$$

$$\boldsymbol{= \lim_{M\to\infty} \sum_{K=1}^{M} \left(\sum_{\substack{L=1 \\ L\neq K}}^{M} \left(\frac{1}{K-L} + \frac{1}{K+L} \right) \left(\frac{4K}{x^2 - K^2\pi^2} \right) \right)}$$

$$\boldsymbol{= \lim_{M\to\infty} \sum_{K=1}^{M} \left(\left(\sum_{L=M-K+1}^{M+K} \frac{1}{L} - \frac{3}{2K} \right) \left(\frac{4K}{x^2 - K^2\pi^2} \right) \right)}$$

$$\boldsymbol{= \lim_{M\to\infty} \sum_{K=1}^{M} \left(\left(\sum_{L=M-K+1}^{M+K} \frac{1}{L} \right) \left(\frac{4K}{x^2 - K^2\pi^2} \right) \right)}$$

$$\boldsymbol{+ \lim_{M\to\infty} \sum_{K=1}^{M} \left(\left(- \frac{3}{2K} \right) \left(\frac{4K}{x^2 - K^2\pi^2} \right) \right)}$$

$$f^2(x) - f'(x) \;-\; \sum_{k=1}^{\infty} \frac{6}{x^2 - k^2\pi^2}$$

$$= \lim_{M\to\infty} \sum_{K=1}^{M} \left(\sum_{L=M-K+1}^{M+K} \frac{1}{L} \right) \left(-\frac{4K}{K^2\pi^2} + \frac{4K}{K^2\pi^2} + \frac{4K}{x^2 - K^2\pi^2} \right)$$

$$+ \lim_{M\to\infty} \sum_{K=1}^{M} \frac{-6}{x^2 - K^2\pi^2}$$

$$= \lim_{M\to\infty} \sum_{K=1}^{M} \left(\sum_{L=M-K+1}^{M+K} \frac{1}{L} \right) \left(-\frac{4K}{K^2\pi^2} + \frac{4Kx^2}{K^2\pi^2(x^2 - K^2\pi^2)} \right)$$

$$- \sum_{K=1}^{\infty} \frac{6}{x^2 - K^2\pi^2} \qquad \text{(which cancels above)}$$

We have, after cancellation,

$$f^2(x) - f'(x)$$

$$= \lim_{M\to\infty} \sum_{K=1}^{M} \left(\left(\sum_{L=M-K+1}^{M+K} \frac{1}{L} \right) \left(-\frac{4K}{K^2\pi^2} + \frac{4Kx^2}{K^2\pi^2(x^2 - K^2\pi^2)} \right) \right)$$

It will now be shown that the part depending on x equals 0.

$$\lim_{M\to\infty} \sum_{K=1}^{M} \left(\left(\sum_{L=M-K+1}^{M+K} \frac{1}{L} \right) \left(\frac{4x^2}{K\pi^2(x^2 - K^2\pi^2)} \right) \right) = 0.$$

First, consider the $\sum_{K<aM}$ part of this sum, for any a where $0 < a < 1$.

Since $\frac{1}{L}$ decreases, $\sum_{L=M-K+1}^{M+K} \frac{1}{L}$, as a lower rectangle sum,

is smaller than the corresponding integral $\int_{u=M-K}^{M+K} \frac{1}{u}\ du$,

Since $\frac{1}{L}$ decreases, $\sum_{L=M-K+1}^{M+K} \frac{1}{L}$, as a lower rectangle sum, is smaller than the corresponding integral $\int_{u=M-K}^{M+K} \frac{1}{u}\; du$, which equals $\ln \frac{M+K}{M-K}$ and, since $K \leq aM$, is thus smaller than $\ln \frac{M+aM}{M-aM} = \ln \frac{1+a}{1-a} = 2 \operatorname{arctanh} a$.

We then have

$$\sum_{K \leq aM} \left(\sum_{L=M-K+1}^{M+K} \frac{1}{L} \right) \left(\frac{4x^2}{K\pi^2(x^2 - K^2\pi^2)} \right)$$

$$\leq 2 \operatorname{arctanh} a \sum_{K=1}^{\infty} \left(\frac{4x^2}{K\pi^2(x^2 - K^2\pi^2)} \right),$$

which can be made smaller than $\frac{\epsilon}{2}$ for any $\epsilon > 0$, if a is small enough.

Next, consider the remaining part, where $aM \leq K \leq M$.

For such K, $\displaystyle\sum_{L=M-K+1}^{M+K} \frac{1}{L} \leq \sum_{L=1}^{2M} \frac{1}{L} < \ln(2M+1)$

$$\sum_{K=aM}^{M} \left(\sum_{L=M-K+1}^{M+K} \frac{1}{L} \right) \left(\frac{4x^2}{K\pi^2(x^2 - K^2\pi^2)} \right)$$

$$< \frac{\ln(2M+1)}{aM\pi^2} \sum_{K=1}^{\infty} \frac{4x^2}{(x^2 - K^2\pi^2)},$$

which can also be made smaller than $\dfrac{\epsilon}{2}$, if M is large enough,

$$\sum_{K=1}^{M} \leq \sum_{K=1}^{aM} + \sum_{K=aM}^{M} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since this can happen with any $\epsilon > 0$, we have $\displaystyle\lim_{M\to\infty} \sum_{K=1}^{M} = 0.$

We are now left with

$$f^2(x) - f'(x) = -\lim_{M\to\infty} \sum_{K=1}^{M} \left(\sum_{L=M-K+1}^{M+K} \frac{1}{L} \right) \frac{4K}{K^2\pi^2} \ ,$$

which converges to a negative constant, say $-C^2$.

Why?

Because the left side is finite for $x \neq n\pi$,

the right side converges for these x's.

Because the right side has no x's, it must be a constant.

Because of the minus sign, it cannot be positive.

If $-C^2$ were equal to 0, we would have $f^2(x) - f'(x) = 0$.

$f'(x) = f^2(x) \geq 0$ would contradict

$$f'(x) = -\sum_{k=-\infty}^{\infty} \frac{1}{(x-k\pi)^2} < 0.$$

$$f'(x) = -f^2(x) - C^2,$$

$$\int \frac{f'(x)}{f^2(x) + C^2}\ dx = -\int 1\ dx,$$

$$\frac{1}{C}\int \frac{\frac{f'(x)}{C}}{\left(\frac{f(x)}{C}\right)^2 + 1}\ dx = -\int 1\ dx,$$

$$\frac{1}{C}\arctan\left(\frac{f(x)}{C}\right) = -x + D,$$

$$f(x) = C\tan\big(C(-x + D)\big)$$

$$f(x) = -C\tan\big(C(x - D)\big)$$

$$
\begin{aligned}
\tan x \;\; &\to \pm\infty \text{ at} & x =& \quad \pm\frac{\pi}{2}, & \pm\frac{3\pi}{2}, && \pm\frac{5\pi}{2}, \ldots \\
\tan C(x-D) &\to \pm\infty \text{ at } & C(x-D) =& \quad \pm\frac{\pi}{2}, & \pm\frac{3\pi}{2}, && \pm\frac{5\pi}{2}, \ldots \\
\tan C(x-D) &\to \pm\infty \text{ at} & x-D =& \quad \pm\frac{\pi}{2C}, & \pm\frac{3\pi}{2C}, && \pm\frac{5\pi}{2C}, \ldots \\
\tan C(x-D) &\to \pm\infty \text{ at} & x =& D\pm\frac{\pi}{2C}, & D\pm\frac{3\pi}{2C}, && D\pm\frac{5\pi}{2C}, \ldots
\end{aligned}
$$

These values of x are spaced at a distance of $\frac{\pi}{C}$.

$$f(x) = \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{x-k\pi} + \frac{1}{x+k\pi}\right) \to \infty \text{ at } x = 0, \pm\pi, \;\; \pm 2\pi,$$

These values of x are spaced at a distance of π.

C equals 1.

$\tan(x - D) \to \pm\infty$ when, and only when,
x approaches the numbers $D \pm \frac{\pi}{2},\;\; D \pm \frac{3\pi}{2},\;\; D \pm \frac{5\pi}{2}, \ldots$

$$f(x) = \frac{1}{x} \;\; + \sum_{k=1}^{\infty} \left(\frac{1}{x - k\pi} + \frac{1}{x + k\pi} \right) \to \infty$$

when, and only when,
x approaches the numbers $0, \pm\pi,\;\; \pm 2\pi,\;\; \pm 3\pi, \ldots$

These two sets of numbers will coincide iff
D equals any of $\pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots.$

For any of these values of D, this would yield
$f(x) = -C\tan\big(C(x - D)\big) = \tan\left(D - x\right) = \cot x.$

$$\boxed{\cot x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2\pi^2} = \frac{1}{x} + \sum_{k=1}^{\infty} \left(\frac{1}{x - k\pi} + \frac{1}{x + k\pi} \right)}$$

For each interval $n\pi \leq x \leq (n+1)\pi$, match antiderivatives of

$$\cot x - \frac{1}{x} = \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2\pi^2} = \sum_{k=1}^{\infty} \frac{-2x}{k^2\pi^2 - x^2} = \sum_{k=1}^{\infty} \frac{-2\left(\frac{x}{k\pi}\right)\frac{1}{k\pi}}{1 - \left(\frac{x}{k\pi}\right)^2}$$

to get $\ln|\sin x| - \ln|x| = \sum_{k=1}^{\infty} \ln\left|1 - \left(\frac{x}{k\pi}\right)^2\right| +$ **a constant,**

$$\frac{\sin x}{x} = C_n \prod_{k=1}^{\infty} \left(1 - \left(\frac{x}{k\pi}\right)^2\right), \text{ where } C_n \neq 0,$$

$$\frac{\sin x}{x} = C_n \prod_{k=1}^{\infty} \left(\left(1 + \frac{x}{k\pi}\right)\left(1 - \frac{x}{k\pi}\right)\right).$$

Taking $\lim_{x \to 0+}$ yields $C_0 = 1$. Taking $\lim_{x \to 0-}$ yields $C_{-1} = 1$.

For $n > 0$, $\quad C_n = \lim\limits_{x \to n\pi+} \dfrac{\dfrac{\sin x}{x}}{\prod\limits_{k=1}^{\infty}\left(\left(1+\dfrac{x}{k\pi}\right)\left(1-\dfrac{x}{k\pi}\right)\right)}$

$$= \lim_{x \to n\pi+} \frac{\sin x}{\left(1-\dfrac{x}{n\pi}\right)} \; \frac{1}{2n\pi \prod\limits_{\substack{k=1 \\ k \neq n}}^{\infty}\left(\left(1+\dfrac{n\pi}{k\pi}\right)\left(1-\dfrac{n\pi}{k\pi}\right)\right)}.$$

Similarly, $C_{n-1} = \lim\limits_{x \to n\pi-}$ **(same stuff). Each** C_n **equals** C_{n-1}**.**

All C_n**'s are equal, and they equal 1.**

This gives us the Weierstrass Product for all x**:**

$$\boxed{\sin x = x \prod_{k=1}^{\infty}\left(1-\frac{x}{k\pi}\right)\left(1+\frac{x}{k\pi}\right) = x \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)}$$

On the entire interval $0 < x < \pi$, in particular,

$$\frac{1}{x} - \cot x \quad \text{equals} \quad -\sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2\pi^2} = \sum_{k=1}^{\infty} \frac{2x}{k^2\pi^2 - x^2}.$$

$$= \sum_{k=1}^{\infty} \frac{2x}{k^2\pi^2} \frac{1}{1 - \dfrac{x^2}{k^2\pi^2}} = \sum_{k=1}^{\infty} \frac{2x}{k^2\pi^2} \left(\sum_{n=0}^{\infty} \left(\frac{x^2}{k^2\pi^2} \right)^n \right)$$

This is a series of positive terms which converges absolutely.

If rearranged, it will converge to the same sum:

$$\frac{1}{x} - \cot x = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{(k\pi)^{2n+2}} = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{2x^{2n+1}}{(k\pi)^{2n+2}}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{2}{(k\pi)^{2n+2}} \right) x^{2n+1} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{2}{(k\pi)^{2n}} \right) x^{2n-1},$$

which is a power series converging for $-\pi < x < \pi$.

We now compare the series convergent for $-\pi < x < \pi$,

$$\cot x - \frac{1}{x} = -\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{2}{(k\pi)^{2n}}\right)x^{2n-1}$$

with the earlier result $\cot x - \frac{1}{x} \simeq \sum_{n=1}^{\infty}\left(\frac{(-1)^n 4^n}{(2n)!}B_{2n}\right)x^{2n-1}$,

which we saw converging at least for $-2 < x < 2$.

Their sums will have the same values, derivatives,

second derivatives, third derivatives, etc., at $x = 0$,

so their coefficients all match: $\sum_{k=1}^{\infty}\frac{2}{(k\pi)^{2n}} = \frac{(-1)^{n-1}4^n}{(2n)!}B_{2n}$,

and these power series will be identical for $-\pi < x < \pi$.

We have the identities $\boxed{\sum_{k=1}^{\infty}\frac{1}{k^{2n}} = \frac{(-1)^{n-1}2^{2n-1}\pi^{2n}}{(2n)!}B_{2n}}$

for $n = 1, 2, \ldots$

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{k^{2n}} &= \frac{(-1)^{n-1}2^{2n-1}\pi^{2n}}{(2n)!}B_{2n} \\
\sum_{k=1}^{\infty}\frac{1}{k^{2}} &= \frac{(-1)^{1-1}2^{2-1}\pi^{2}}{(2)!}B_{2} &&= \frac{(-1)^{0}2^{1}\pi^{2}}{2}\ \frac{1}{6} &&= \frac{\pi^{2}}{6} \\
\sum_{k=1}^{\infty}\frac{1}{k^{4}} &= \frac{(-1)^{2-1}2^{4-1}\pi^{4}}{4!}B_{4} &&= \frac{(-1)2^{3}\pi^{4}}{24}\ \frac{-1}{30} &&= \frac{\pi^{4}}{90} \\
\sum_{k=1}^{\infty}\frac{1}{k^{6}} &= \frac{(-1)^{3-1}2^{6-1}\pi^{6}}{6!}B_{6} &&= \frac{(-1)^{2}2^{5}\pi^{6}}{720}\ \frac{1}{42} &&= \frac{\pi^{6}}{945} \\
\sum_{k=1}^{\infty}\frac{1}{k^{8}} &= \frac{(-1)^{4-1}2^{8-1}\pi^{8}}{8!}B_{8} &&= \frac{(-1)2^{7}\pi^{8}}{40320}\ \frac{-1}{30} &&= \frac{\pi^{8}}{9450} \\
\sum_{k=1}^{\infty}\frac{1}{k^{10}} &= \frac{(-1)^{5-1}2^{9}\pi^{10}}{10!}B_{10} &&= \frac{(-1)^{4}2^{9}\pi^{10}}{362880}\ \frac{5}{66} &&= \frac{\pi^{10}}{93555}
\end{aligned}
$$

$$
\begin{aligned}
1 < \sum_{k=1}^{\infty} \frac{1}{k^{2n}} &= 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \frac{1}{7^{2n}} + \dots \\
&< 1 + \frac{1}{2^{2n}} + \frac{1}{2^{2n}} + \frac{1}{4^{2n}} + \frac{1}{4^{2n}} + \frac{1}{4^{2n}} + \frac{1}{4^{2n}} + \dots \\
&= 1 + \frac{2}{2^{2n}} + \frac{4}{4^{2n}} + \frac{8}{8^{2n}} + \dots \\
&= 1 + \frac{1}{2^{2n-1}} + \frac{1}{4^{2n-1}} + \frac{1}{8^{2n-1}} + \dots = \frac{1}{1 - \frac{1}{2^{2n-1}}}
\end{aligned}
$$

yields both the inequalities $1 < \sum_{k=1}^{\infty} \frac{1}{k^{2n}} < \frac{1}{1 - \frac{1}{2^{2n-1}}} \leq 2,$

and the limit $\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = 1.$

For the Bernoulli numbers the corresponding inequalities are

$$1 < \frac{(-1)^{n-1}2^{2n-1}\pi^{2n}}{(2n)!}B_{2n} < \frac{1}{1-\dfrac{1}{2^{2n-1}}} \leq 2$$

$$\frac{(2n)!}{2^{2n-1}\pi^{2n}} < \left|B_{2n}\right| = (-1)^{n-1}B_{2n} < \frac{(2n)!}{\pi^{2n}\left(2^{2n-1}-1\right)} \quad \text{(sharper)}$$

$$\frac{(2n)!}{2^{2n-1}\pi^{2n}} < \left|B_{2n}\right| = (-1)^{n-1}B_{2n} < \frac{4(2n)!}{2^{2n}\pi^{2n}} \quad \text{(simpler)}$$

with the asymptotic formulas $\boxed{B_{2n} \sim (-1)^{n-1}\dfrac{(2n)!}{2^{2n-1}\pi^{2n}}}$

and $\boxed{B_{2n} \sim (-1)^{n-1}\dfrac{4n^{2n+\frac{1}{2}}}{e^{2n}\pi^{2n-\frac{1}{2}}}}$

Now we shall consider the function $g(x,t) = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n$.

For each value of x, this is a power series in t which converges for each t satisfying $|t| < 2\pi$.

$$\begin{aligned}
\frac{d}{dx} g(x,t) &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{B_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{B'_n(x)}{n!} t^n \\
&= \sum_{n=0}^{\infty} \frac{nB_{n-1}(x)}{n!} t^n = \sum_{n=1}^{\infty} \frac{nB_{n-1}(x)}{n(n-1)!} t^n \\
&= \sum_{n=1}^{\infty} \frac{B_{n-1}(x)}{(n-1)!} t^n = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^{k+1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^{n+1} \\
&= t \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = t g(x,t), \quad \textbf{for each value of } t.
\end{aligned}$$

For each constant value of t, we have $\dfrac{d}{dx}g(x,t) = tg(x,t)$

and
$$\begin{aligned} \frac{d}{dx}\left(g(x,t)e^{-xt}\right) &= \left(\frac{d}{dx}g(x,t)\right)e^{-xt} + g(x,t)\frac{d}{dx}e^{-xt} \\ &= \left(tg(x,t)\right)e^{-xt} + g(x,t)te^{-xt} \\ &= 0. \end{aligned}$$

As a function of x, $g(x,t)e^{-xt}$ must be a constant,
which depends on the value of t:
$g(x,t)e^{-xt}$ equals $C(t)$ for some function C.

We now have $\displaystyle\sum_{n=0}^{\infty} \frac{B_n(x)}{n!}t^n = g(x,t)$ equal to $C(t)e^{xt}$,
for all x and t satisfying $|t| < 2\pi$.

Starting from $C(t)e^{xt} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n$,

we keep t constant and integrate these functions of x:

$$\begin{aligned}
\int_{x=0}^{1} C(t)e^{xt}\ dx &= \int_{x=0}^{1} \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n\ dx, \\
C(t)\int_{x=0}^{1} e^{xt}\ dx &= \sum_{n=0}^{\infty} \frac{\int_{x=0}^{1} B_n(x)\ dx}{n!} t^n, \\
C(t)\frac{e^{xt}}{t}\bigg|_{x=0}^{1} &= \quad \frac{\int_{x=0}^{1} B_0(x)\ dx}{0!} t^0, \\
C(t)\frac{e^t - e^0}{t} &= \ 1, \\
C(t) &= \ \frac{t}{e^t - 1}, \text{ for } 0 < |t| < 2\pi.
\end{aligned}$$

A Generating Function for Bernoulli Polynomials

$$\boxed{\text{For } 0 < |t| < 2\pi, \quad \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{te^{xt}}{e^t - 1}}$$

A Generating Function for Periodic Bernoulli Functions

$$\boxed{\text{For } 0 < |t| < 2\pi, \quad \sum_{n=0}^{\infty} \frac{\hat{B}_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{B_n(\{x\})}{n!} t^n = \frac{te^{\{x\}t}}{e^t - 1}}$$

A Generating Function for Bernoulli Numbers

$$\boxed{\text{For } 0 < |t| < 2\pi, \quad \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1}}$$

To justify the $\frac{d}{dx}\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n = \sum_{n=0}^{\infty}\frac{d}{dx}\frac{B_n(x)}{n!}t^n$ step above, check the pattern

$$g(u) \;=\; g(x) \;+\; g'(x)(u-x) \;+\; o(u-x):$$

$$\frac{B_n(u)}{n!} \;=\; \frac{B_n(x)}{n!} \;-\; \frac{B_n'(x)}{n!}(u-x) \;+\; \frac{B_n^{(2)}(c_n)(u-x)^2}{2n!},$$

$$\frac{B_n(u)}{n!} \;=\; \frac{B_n(x)}{n!} \;-\; \frac{B_{n-1}(x)}{(n-1)!}(u-x) \;+\; \frac{B_{n-2}(c_n)(u-x)^2}{2(n-2)!},$$

where each c_n lies between u and x.

$$\sum_{n=0}^{\infty}\frac{B_n(u)}{n!}t^n \;=\; \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n \;-\; \sum_{n=0}^{\infty}\frac{B_{n-1}(x)}{(n-1)!}(u-x)t^n$$
$$+\;\frac{1}{2}\sum_{n=0}^{\infty}\frac{B_{n-2}(c_n)}{(n-2)!}t^n(u-x)^2.$$

$\frac{1}{2}\sum_{n=0}^{\infty}\frac{B_{n-2}(c_n)}{(n-2)!}t^n(u-x)^2$ will be $o(u-x)$

if $\sum_{n=0}^{\infty}\frac{B_{n-2}(c_n)}{(n-2)!}t^n$ converges,

and it does, for $0 \le x \le 1$ and $0 \le u \le 1$,

since it converges absolutely there:

$\sum_{n=0}^{\infty}\left|\frac{B_{n-2}(c_n)}{(n-2)!}t^n\right|$ converges by the root test:

$$\text{for even } n, \quad \sqrt[n]{\left|\frac{B_{n-2}(c_n)}{(n-2)!}t^n\right|} \sim \frac{|t|}{2\pi} < 1.$$

If x and u belong to any other bounded interval,

the inequalities are similar.

We can return to the cothangent and the cotangent from here.

For $0 < |t| < 2\pi$, $$\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1} \ .$$

For $0 < |x| < \pi$, $$\sum_{n=0}^{\infty} \frac{B_n}{n!} (2x)^n = \frac{2x}{e^{2x} - 1} \ ,$$

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} (2x)^{n-1} = \frac{1}{e^{2x} - 1} \ ,$$

$$\sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{B_n}{n!} (2x)^{n-1} + \underbrace{\frac{B_1}{1!} (2x)^{1-1}}_{= \ -\frac{1}{2}} = \frac{1}{e^{2x} - 1} \ ,$$

$$\sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{B_n}{n!} (2x)^{n-1} = \frac{1}{e^{2x} - 1} + \frac{1}{2} \ ,$$

$$
\begin{aligned}
\textbf{For } 0 < |x| < \pi, \qquad \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{B_n}{n!}(2x)^{n-1} &= \frac{1}{e^{2x}-1} + \frac{1}{2} \ , \\
\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} 2^{2n-1} x^{2n-1} &= \frac{e^{2x}+1}{2(e^{2x}-1)} \ , \\
\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} 2^{2n} x^{2n-1} &= \frac{e^{2x}+1}{e^{2x}-1} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}} \ , \\
\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} 4^{n} x^{2n-1} &= \coth x \ , \\
i \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} 4^{n} (ix)^{2n-1} &= i \coth ix \ , \\
\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} 4^{n} (-1)^{n} (x)^{2n-1} &= \cot x \ .
\end{aligned}
$$