Math 290 Section 6 — Final Exam
Final Exam Week – April 2014 – Testing Center
Professor: David Cardon, 326 TMCB, Campus Ext. 2-4863

Instructions:

• Questions 1–10 are true-false worth 2 points each. Mark A for true, B for false.

• Questions 11–20 are multiple choice worth 3 points each. Mark the correct answer on your bubble sheet.

• Questions 21–25 are written response questions worth 10 points each. Neatly write your solutions directly on the exam paper. To receive full credit you must provide complete and correct explanations.

• In written response questions words like find, show, solve, determine, or prove mean that you should give complete explanations of the reasoning involved in the finding, showing, solving, determining, or proving.

• Notes, books, and calculators are not allowed.

• No time limit.

For instructor use only:

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True-False Questions

1. If \( A, B, \) and \( C \) are any sets, then \( A - (B \cup C) = (A - B) \cup (A - C) \).

2. The statement “\( P \Rightarrow Q \)” is logically equivalent to the statement “\( \sim (\sim P) \lor Q \)”.

3. If \( x, y \in \mathbb{R} \) then \( |x + y| \geq |x| + |y| \).

4. The negation of the statement “There exist irrational numbers \( a, b \) such that \( a^b \) is rational.” is the statement “For all irrational numbers \( a, b \), it is true that \( a^b \) is irrational.”

5. Let \( R \) be an equivalence relation on a nonempty set \( A \), and denote the equivalence class of \( a \in A \) by \([a]\). For \( a, b \in A \), we have \( a \sim b \) if and only if \([a] = [b]\).

6. Let \( A \) be a nonempty set, and suppose \( f : A \rightarrow A \) is a surjective function. Then \( f \) is injective.

7. Every nonempty subset of the positive integers is well ordered.

8. Let \( S \) and \( T \) be uncountable sets and suppose \( T \subseteq S \). Then \(|T| = |S|\).

9. If \( a \) and \( b \) are odd integers, then \( 4 \mid (a - b) \) or \( 4 \mid (a + b) \).

10. Let \( a, b, c \in \mathbb{Z}, \) with \( a \neq 0 \). If \( a \mid bc \) and \( a \nmid b \), then \( a \mid c \).

Multiple Choice Questions

11. Which of the following definitions is incorrect?

   (a) An equivalence relation is a relation on a nonempty set which is reflexive, symmetric and transitive.

   (b) A function \( f : A \rightarrow B \) is injective if for every \( a_1, a_2 \in A \), \( f(a_1) \neq f(a_2) \) implies that \( a_1 \neq a_2 \).

   (c) Two sets \( A \) and \( B \) are numerically equivalent if there is a bijection \( f : A \rightarrow B \).

   (d) A partition of a set \( A \) is a collection of nonempty pairwise disjoint subsets of \( A \) whose union is \( A \).

   (e) For integers \( a \) and \( b \) where \( a \neq 0 \) we say that \( a \mid b \) if there exists an integer \( c \) such that \( ac = b \).

   (f) None of the above. All of these definitions are correct.

12. Which of the following would be the most effective method for proving the statement \( P \Rightarrow (Q \lor R) \)?

   (a) Assume \( P \) and \( \sim Q \) and prove that \( R \) is true.

   (b) Assume \( P \) and \( Q \) and prove that \( R \) is true.

   (c) Assume \( P \) and \( Q \) and prove that \( R \) is false.

   (d) Assume \( \sim Q \) or \( \sim R \) and prove \( \sim P \).

   (e) Assume \( Q \) and \( R \) and prove that \( P \) is true.
13. Which of the following is the contrapositive of the statement “If every $x \in S$ is prime, then every $x \in S$ is odd.”?

(a) If there is an $x \in S$ which is prime, then there is an $x \in S$ that is odd.
(b) If there is an $x \in S$ that is odd, then there is an $x \in S$ that is prime.
(c) If there is an $x \in S$ that is even, then there is an $x \in S$ that is prime.
(d) If every $x \in S$ is odd, then every $x \in S$ is prime.
(e) If every $x \in S$ is not prime, then every $x \in S$ is odd.
(f) If every $x \in S$ is odd, then there is an $x \in S$ that is prime.
(g) If every $x \notin S$ is even, then every $x \notin S$ is not prime.

14. Which of the following statements is true?

(a) Let $x \in \mathbb{Z}$. If $4x + 7$ is odd, then $x$ is even.
(b) There exists a real number $x$ such that $x^2 < x < x^3$.
(c) Let $x, y, z \in \mathbb{Z}$. If $z = x - y$ and $z$ is even, then $x$ and $y$ are odd.
(d) Every odd integer is a sum of four odd integers.
(e) If $x \in \mathbb{Z}$ is odd, then $x^2 + x$ is even.
(f) For every two sets $A$ and $B$, $(A \cup B) - B = A$.

15. Which of the following is not an equivalence relation on the set $\mathbb{Z}$ of integers?

(a) $x R y$ if $7 | (x - y)$.
(b) $x R y$ if $6 | (x^2 - y^2)$
(c) $x R y$ if $x + y \geq 0$.
(d) $x R y$ if $x + y = 2x$.
(e) $x R y$ if $x^2 - 2xy + y^2 \geq 0$.

16. Which of the following is the negation of the statement “$\lim_{x \to a} f(x) = L$.”?

(a) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.
(b) There exists an $\epsilon \leq 0$ such that for all $\delta \leq 0$, there is an $x \in \mathbb{R}$ with $0 \geq |x - a| \geq \delta$ such that $|f(x) - L| \geq \epsilon$.
(c) There exists an $\epsilon > 0$ such that for all $\delta > 0$, there is an $x \in \mathbb{R}$ with $0 < |x - a| < \delta$ such that $|f(x) - L| \geq \epsilon$.
(d) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| \geq \epsilon$. 
17. Evaluate the proposed proof of the following result. Choose the most complete correct answer.

**Theorem:** If two functions \( f: A \rightarrow B \) and \( g: B \rightarrow C \) are bijective, then \( g \circ f \) is bijective.

**Proof.** Suppose that \( f \) and \( g \) are both bijective. Then \( f \) is both injective and surjective, and \( g \) is both injective and surjective.

Let \( c \in C \). Since \( g \) is surjective, \( c = g(b) \) for some \( b \in B \). Since \( f \) is surjective, \( b = f(a) \) for some \( a \in A \). Then \( c = g(b) = g(f(a)) = (g \circ f)(a) \). Hence, \( g \circ f \) is surjective.

Suppose that \( a_1, a_2 \in A \) and \( a_1 = a_2 \). Since \( f \) is injective, \( f(a_1) = f(a_2) \). Since \( g \) is injective, \( g(f(a_1)) = g(f(a_2)) \). Hence \( (g \circ f)(a_1) = (g \circ f)(a_2) \), so \( g \circ f \) is injective.

Since \( g \circ f \) is both injective and surjective, it is bijective.

(a) The theorem and the proof are correct.
(b) The proof is correct but the theorem is false.
(c) The proof does not successfully prove that \( g \circ f \) is surjective.
(d) The proof does not successfully prove that \( g \circ f \) is injective.
(e) The proof proves neither that \( g \circ f \) is surjective, nor that it is injective.
(f) The proof is irrelevant because injectivity and surjectivity have nothing to do with proving a function to be bijective.

18. Which of the following definitions is correct:

(a) The difference of two sets \( A \) and \( B \) is \( (A - B) \cup (B - A) \).
(b) A function \( f: A \rightarrow B \) is injective if every element of \( A \) maps to exactly one element of \( B \).
(c) A relation \( R \) on a nonempty set \( A \) is transitive if for \( a, b, c \in A \), \( aRb \) and \( bRc \) imply \( cRa \).
(d) A set \( A \) is denumerable if it is contained in the set \( \mathbb{N} \) of natural numbers.
(e) A sequence \( \{a_n\} \) diverges to infinity if it does not converge to any finite limit.
(f) None of the above. All of these definitions are incorrect.

19. In an \( \epsilon-\delta \) proof that \( \lim_{x \to 1} 3x + 5 = 8 \), which of the following is the largest \( \delta \) that we can associate with a given \( \epsilon > 0 \)?

(a) \( \delta = 3 \)  (b) \( \delta = 1/3 \)  (c) \( \delta = \epsilon \)  (d) \( \delta = 3\epsilon \)

(e) \( \delta = \epsilon/3 \)  (f) \( \delta = \epsilon/5 \)  (g) \( \delta = \min(\epsilon, 1) \)  (h) \( \delta = 0 \)

20. Let \( A \) be a nonempty set, and let \( R \) be an equivalence relation on \( A \). Let \( E \) be the set of equivalence classes of \( R \) on \( A \), with the equivalence class of \( a \in A \) denoted by \([a]\). Define \( f: A \rightarrow E \) by \( f(a) = [a] \).

Choose the most complete answer below.

(a) \( f \) is not a well defined function.
(b) \( f \) must be an injective function.
(c) \( f \) must be a surjective function.
(d) \( f \) must be a bijective function.
(e) \( f \) may be neither surjective nor injective, but it is a function.
21. Use induction to prove that for every $n \in \mathbb{Z}$ with $n \geq 5$,

$$2^n > n^2.$$
22. Prove the following statement in three different ways: (1) by a direct proof, (2) using the contra-
positive, and (3) using proof by contradiction. Be sure to correctly label which proof uses which

If \( n \) is an even integer, then \( 3n + 7 \) is odd.
23. Let $A, B, C, D$ be sets with $A \subseteq C$ and $B \subseteq D$. Prove or disprove the following statements:

(a) If $A \cap B = \emptyset$, then $C \cap D = \emptyset$.

(b) If $C \cap D = \emptyset$, then $A \cap B = \emptyset$. 

24. Give an $\epsilon$-$\delta$ proof that

\[
\lim_{x \to 4} (x^2 - 2x + 2) = 10.
\]
25. Prove that $(2, 3) \cup (4, 5)$ is numerically equivalent to $\mathbb{R}$.

(You may assume that $(0, 1)$ and $\mathbb{R}$ are numerically equivalent if this helps.)