Instructions:

- Questions 1–10 are true-false worth 2.5 points each. Mark A for true, B for false.
- Questions 11–20 are multiple choice worth 2.5 points each. Mark the correct answer on your bubble sheet.
- Questions 21–25 are written response questions worth 10 points each. Neatly write your solutions directly on the exam paper. *To receive full credit you must provide complete and correct explanations.*
- In written response questions words like *find, show, solve, determine,* or *prove* mean that you should give complete explanations of the reasoning involved in the finding, showing, solving, determining, or proving.
- Notes, books, and calculators are not allowed.
- No time limit.

For instructor use only:

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True-false questions

1. The statement “For any \( x \in \mathbb{R}, \) if \( x^2 + 4x + 4 < 0, \) then \( x^5 - 7x + 3 > 0 \)” has a vacuous proof.

2. Let \( A, B, \) and \( C \) be sets. If \( A \cup B \subseteq B \cap C, \) then \( A \subseteq C. \)

3. For each \( i \in \mathbb{N}, \) let \( A_i \) be a subset of the integers. Suppose that the sets are nested

\[
A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots
\]

(In other words, \( A_2 \) is a subset of \( A_1, \) \( A_3 \) is a subset of \( A_2, \) and so forth.) If each individual \( A_i \) is an infinite set, then the intersection

\[
\bigcap_{i=1}^{\infty} A_i
\]

is also an infinite set.

4. Consider the statements

\( P : \) 17 is even. \quad \( Q : \) 19 is prime.

Determine the truth value of the implication \( (\sim P) \Rightarrow (\sim Q). \)

5. Given \( a, b, c, d, e \in \mathbb{Z} \) with \( a \neq 0, \) if \( a|b \) and \( a|c, \) then \( a|(bd + ce). \)

6. Let \( A \) and \( B \) be sets. If \( A \subseteq B, \) then \( A \times A \subseteq B \times B. \)

7. Let \( x \in \mathbb{Z}. \) A proof of the statement “If \( x^2 \) is even, then \( x \) is even.” may begin with the sentence “Assume \( x \) is odd.”

8. The negation of the statement

“There exists \( x \in \mathbb{R} \) such that if \( x^2 - 1 < 0, \) then \( x < 1/2. \)”

is the statement

“For all \( x \in \mathbb{R}, \) we have \( x \geq 1/2. \)”

9. If \( a \equiv 3 \pmod{4} \) and \( b \equiv 2 \pmod{4}, \) then \( a + b \equiv 9 \pmod{4}. \)

10. Let

\( “P(n): (5n - 6)/3 \) is an integer.”

be an open sentence over the domain \( \mathbb{Z}. \) Determine the truth value of the quantified statement:

\( \exists n \in \mathbb{Z}, P(n). \)
Multiple choice questions

11. The statement “For all positive real numbers $x$ and $y$, if $x < y$, then $x^2 < y^3$.” is:
(a) True, vacuously.  (b) True, trivially.
(c) True, by a direct proof.  (d) False, by a counter-example.
(e) False, because not all real numbers are positive.  (f) None of the above.

12. Consider the following sets:

\[ A = \{ x \in \mathbb{N} : x \leq 4 \} \]
\[ B = \{ r \in \mathbb{R} : 0 < r < 5 \} \]
\[ C = \{ y \in \mathbb{Z} : (y - 1)(y - 2)(y - 3)(y - 4) = 0 \} \]
\[ D = \{ 1, 2, 3, 4 \} \]

(a) All the sets are equal.  (b) Only $A$, $B$, and $C$ are equal.
(c) Only $A$, $B$, and $D$ are equal.  (d) Only $A$, $C$, and $D$ are equal.
(e) Only $B$, $C$, and $D$ are equal.  (f) Only $A$ and $B$ are equal.
(g) Only $A$ and $C$ are equal.  (h) Only $A$ and $D$ are equal.
(i) None of the above.


**Corollary:** Let $d$ be a square-free integer. If $p \in \mathbb{Z}[\sqrt{d}]$ and $N(p)$ is a prime integer in $\mathbb{Z}$, then $p$ is irreducible in $\mathbb{Z}[\sqrt{d}]$.

Suppose we know that $d$ is a square-free integer and that $p \in \mathbb{Z}[\sqrt{d}]$. Which of the following must be true?

(a) If $p$ is not irreducible in $\mathbb{Z}[\sqrt{d}]$, then $N(p)$ is not a prime integer in $\mathbb{Z}$.
(b) If $p$ is irreducible in $\mathbb{Z}[\sqrt{d}]$, then $N(p)$ is a prime integer in $\mathbb{Z}$.
(c) If $p$ is not irreducible in $\mathbb{Z}[\sqrt{d}]$, then $N(p)$ is a prime integer in $\mathbb{Z}$.
(d) If $p$ is irreducible in $\mathbb{Z}[\sqrt{d}]$, then $N(p)$ is not a prime integer in $\mathbb{Z}$.
(e) $p$ is not irreducible in $\mathbb{Z}[\sqrt{d}]$ and $N(p)$ is a prime integer in $\mathbb{Z}$.
14. Suppose that you wish to prove a statement of the form “If $P$, then $(Q$ and $R)$.” Which of the following would not be a good way to begin?

(a) Assume $P$.
(b) Assume $P \land ((\sim Q) \lor (\sim R))$.
(c) Assume $P \lor (\sim Q)$.
(d) Assume $\sim (Q \land R)$.
(e) None of the above; all of these are acceptable ways to begin a valid proof.

15. If $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$, and $C = \{2, 4, 5\}$, then how many elements does $(A \cup C) - B$ have?

(a) 0  (b) 1  (c) 2  (d) 3  (e) 4  (f) 5  (g) 6  (h) 7  (i) 8  (j) 9

16. Let $P$ and $Q$ be statements. How many of the following statements are tautologies?

$P \land (\sim P)$, $P \Rightarrow Q$, $(P \land (P \Rightarrow Q)) \Rightarrow Q$, $(P \Rightarrow Q) \Leftrightarrow ((\sim Q) \Rightarrow (\sim P))$

(a) None  (b) One  (c) Two  (d) Three  (e) All four

17. The statement “If $x^2 \in S$, then $x \in P$” is to be proved. Suppose an attempted proof begins with the sentence “Assume that $x^2 \notin S$.” Which of the following best describes the proof?

(a) The proof is a direct proof.
(b) The proof is a proof by contrapositive.
(c) The proof is a proof by contradiction.
(d) The proof will show the statement is vacuously true.
(e) None of the above; no correct proof can begin with that sentence.
18. Consider the problem “Prove that there exists a real number whose cube equals its square.”

Consider the following three potential proofs:

1. We need to solve \( x^3 = x^2 \). Thus, \( x^2(x - 1) = 0 \), whose solutions are \( x = 0 \) and \( x = 1 \). Plugging these back in, we see that both are solutions.

2. The statement is true because \( 1^3 = 1^2 \).

3. Let \( f(x) = x^3 - x^2 \). Since \( f \) is a polynomial, it is continuous. Now \( f(-1) = -2 < 0 \) and \( f(2) = 4 > 0 \), so by the intermediate value theorem, there must be some \( c \in (-1, 2) \) such that \( f(c) = 0 \), so that \( c^3 = c^2 \).

Which of the following statements is correct?

(a) Proof 3 is the only one of the three proofs that is correct.
(b) Proofs 1 and 2 are correct, but not 3 because it does not give an explicit solution.
(c) Proofs 1 and 3 are correct because they allow both solutions \( x = 0 \) and \( x = 1 \), but proof 2 is incorrect because it only gives one of the solutions.
(d) Only proof 2 is correct, because the others introduce unnecessary information.
(e) All three proofs are correct.

19. Evaluate the proof of the following result.

**Result:** If \( m \) is an even integer and \( n \) is an odd integer, then \( 3m + 5n \) is odd.

**Proof.** Let \( m \) be an even integer and \( n \) an odd integer. Then \( m = 2k \) and \( n = 2k + 1 \), where \( k \in \mathbb{Z} \). Therefore,

\[
3m + 5n = 3(2k) + 5(2k + 1) = 6k + 10k + 5 \\
= 16k + 5 = 2(8k + 2) + 1.
\]

Since \( 8k + 2 \) is an integer, \( 3m + 5n \) is odd. \( \square \)

Which of the following is correct?

(a) The proof is correct.
(b) The proof proves the converse of the statement.
(c) The proof proves the contrapositive of the statement.
(d) The proof contains arithmetic mistakes, which makes it incorrect.
(e) The proof contains an extra assumption, which makes it incorrect.
20. Consider the following collections of subsets of the set $A = \{1, 2, 3, 4, 5, 6\}$:

$S_1 = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\}$
$S_2 = \{\{1, 2, 3\}, \{4\}, \emptyset, \{5, 6\}\}$
$S_3 = \{\{1, 2\}, \{3, 4, 5\}, \{5, 6\}\}$
$S_4 = \{\{1, 4\}, \{3, 5\}, \{2\}\}$.

Which of these sets are partitions of $A$? (Choose the most complete answer.)

(a) $S_1$  
(b) $S_2$  
(c) $S_3$  
(d) $S_4$  
(e) $S_1$ and $S_2$  
(f) $S_1$ and $S_4$  
(g) $S_2$ and $S_3$  
(h) $S_1$ and $S_2$ and $S_4$  
(i) None of the above.
21. Use a truth table to determine whether the statements:

\[ P \implies (Q \lor R) \]

and

\[ (P \lor (\sim Q)) \implies R \]

are logically equivalent. Be sure to indicate at the end whether the statements are equivalent or not.

\[ \begin{array}{ccc}
P & Q & R \\
\hline
\end{array} \]
22. Prove that if $n$ is an even integer, then $3n + 2$ is even in each of the following three ways:

(i) a direct proof,
(ii) a proof by contrapositive,
(iii) a proof by contradiction.

(Do not assume any facts about odd and even numbers except the definitions. In particular, do not cite results proved in the book.)
23. Let $x$ be a real number. Prove that $x^2 < x$ if and only if $0 < x < 1$. 
24. Let $A$, $B$, and $C$ be sets. Prove that

$$(A - B) \cap (A - C) = A - (B \cup C).$$

(Give justification for each step. Do not use a Venn diagram as your proof.)
25. Give a partition of \( \mathbb{N} \) (the set of all positive integers) into five parts, with three of the parts each containing 4 elements, and one of the other two parts being infinite.