Math 290 Section 6 — Midterm Exam 3
Tue, Apr 1 through Thu, Apr 3 — Testing Center
Professor: David Cardon, 326 TMCB, Campus Ext. 2-4863

Instructions:

• Questions 1–10 are true-false worth 2 points each. Mark A for true, B for false.

• Questions 11–20 are multiple choice worth 3 points each. Mark the correct answer on your bubble sheet.

• Questions 21–25 are written response questions worth 10 points each. Neatly write your solutions directly on the exam paper. To receive full credit you must provide complete and correct explanations.

• In written response questions words like find, show, solve, determine, or prove mean that you should give complete explanations of the reasoning involved in the finding, showing, solving, determining, or proving.

• Notes, books, and calculators are not allowed.

• No time limit.

For instructor use only:

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True-False Questions

\( \mathbf{F} \) 1. The remainder when dividing 73 by 19 is 3.

\( \mathbf{T} \) 2. Let \( a, b, q, r \in \mathbb{N} \). If \( b = aq + r \) then \( \gcd(a, b) = \gcd(a, r) \).

\( \mathbf{F} \) 3. The Schröder–Bernstein Theorem is the statement that \( |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| \).

\( \mathbf{F} \) 4. The numbers \( 3n \) and \( 5n + 4 \) are relatively prime for all \( n \in \mathbb{N} \).

\( \mathbf{T} \) 5. The set \( \{a + b : a \in \mathbb{N}, b \in \mathbb{N}, a + b \leq 5\} \) is countable.

\( \mathbf{T} \) 6. Every uncountable set contains a denumerable set.

\( \mathbf{T} \) 7. \( |\mathbb{Q} - \mathbb{N}| = \aleph_0 \).

\( \mathbf{T} \) 8. Let \( A \) and \( B \) be nonempty sets such that \( B \subseteq A \). If there exists an injective function from \( A \) to \( B \), then there exists a bijective function from \( A \) to \( B \).

\( \mathbf{F} \) 9. Let \( A \) be a nonempty set and let \( B = \{0, 1\} \). Then \( |A| < |A \times B| \).

\( \mathbf{F} \) 10. Let \( p \) be a positive integer. If \( \sqrt{p} \) is irrational, then \( p \) is prime.

Multiple Choice Questions

11. Suppose that \( f : A \to B \) is an injective map between sets \( A \) and \( B \). How many of the following statements would imply that \( |A| = |B| \)?

- \( A \) is uncountable.
- \( B \) is denumerable.
- \( B \) is a subset of \( A \).
- There is an injective map from \( B \) to \( A \).
- There is a bijection from \( A \) to \( B \).

(a) One  
(b) Two  
(c) Three  
(d) Four  
(e) Five

12. Let \( n \) be a positive integer. Which of the following pairs of numbers must be relatively prime?

(a) \( 2n \) and \( 4n + 5 \).
(b) \( 2n + 1 \) and \( 3n + 5 \).
(c) \( 3n + 4 \) and \( 2n + 3 \).
(d) \( 2n + 5 \) and \( 3n + 1 \).
13. Let $a$, $b$, and $c$ be nonzero integers. The statement

   If $c \mid ab$ and if $a$ and $b$ are relatively prime, then $c \mid a$ or $c \mid b$.

is

(a) vacuously true.
(b) trivially true.
(c) true, by Euclid’s Lemma.
(d) false, by a counterexample.
(e) false, by definition.

14. Let $A = \{a, b, c, d\}$ and consider the following four subsets of $\mathcal{P}(A)$:

   $$A_a = \emptyset, \quad A_b = \{c, d\}, \quad A_c = \{a, b, c\}, \quad A_d = \{b, d\}.$$  

Which of the following sets is $B = \{x \in A : x \not\in A_x\}$?

(a) $\{a, b\}$
(b) $\{b, d\}$
(c) $\{b\}$
(d) $\{b, c, d\}$
(e) $A$

15. Evaluate the following proposed proof by choosing the best answer among the options given.

**Result** The sets $(0, \infty)$ and $[0, \infty)$ are numerically equivalent.

**Proof** Define the function $f : (0, \infty) \rightarrow [0, \infty)$ by $f(x) = x$. First we show that $f$ is one-to-one. Assume $f(a) = f(b)$. Then $a = b$ and so $f$ is one-to-one.

Next, we show that $f$ is onto. Let $r \in [0, \infty)$. Since $f(r) = r$, the function $f$ is onto. Since $f$ is bijective, $|(0, \infty)| = |[0, \infty)|$.

(a) The proof is correct.
(b) The proof that $f$ is onto is correct, but the proof that $f$ is one-to-one contains an error.
(c) The proof that $f$ is one-to-one is correct, but the proof that $f$ is onto contains an error.
(d) The proof that $f$ is one-to-one contains an error, and the proof that $f$ is onto contains an error.

16. Which of the following sets has the largest cardinality?

(a) $\mathbb{Z}$
(b) $\mathcal{P}(\mathbb{N})$
(c) $\mathbb{R}$
(d) $\mathbb{N}$
(e) $\{0, 1\}^\mathbb{R}$
(f) Two of these sets both have the largest cardinality.
17. Which of the following statements is false?

(a) The set of rational numbers is uncountable.
(b) The empty set is countable.
(c) The set of real numbers is uncountable.
(d) The set of irrational numbers is uncountable.

18. Which of the following arguments demonstrates that \( \mathbb{Z} \) is denumerable? Choose the most correct answer.

(a) The map \( f: \mathbb{N} \to \mathbb{Z} \) given by \( f(n) = n \) is an injection. So by the Schröder-Bernstein theorem we have \( |\mathbb{N}| = |\mathbb{Z}| \).
(b) A theorem in the books says that if \( A \subseteq B \) and \( A \) is denumerable then \( B \) is denumerable. Since \( \mathbb{N} \subseteq \mathbb{Z} \), we know \( \mathbb{Z} \) is denumerable.
(c) The map \( g: \mathbb{N} \to \mathbb{Z} \) given by \( g(n) = n/2 \) if \( n \) is even, and \( g(n) = (1 - n)/2 \) if \( n \) is odd is a bijection. Thus \( |\mathbb{N}| = |\mathbb{Z}| \).
(d) The set of integers is not denumerable. It is impossible to prove a false statement.
(e) None of the above.

19. \( \gcd(687, 699) = \)

(a) 1
(b) 3
(c) 5
(d) 7
(e) 9

20. Which of the following statements is true?

(a) If \( A \) is denumerable, then \( |A| = |\mathbb{R}| \).
(b) There exists a surjective function \( f: \mathbb{Q} \to \mathbb{R} \).
(c) If \( A \) is uncountable, then \( |A| = |\mathbb{R}| \).
(d) If \( A, B, \) and \( C \) are sets with \( A \subseteq B \subseteq C \) such that \( A \) and \( C \) are countable, then \( B \) is countable.
(e) If \( A \) is denumerable and \( A \) is a proper subset of \( B \), then \( B \) is uncountable.
(f) None of the above is true.
21. Define the bold-faced terms by completing the sentences.

(a) Two non-empty sets $A$ and $B$ have the **same cardinality** if ...

\[
\text{either both sets are empty or there exists a bijective function } f \text{ from } A \text{ to } B.
\]

(b) A **linear combination** of integers $a, b \in \mathbb{Z}$ is ...

\[
\text{an integer of the form } ax + by \text{ where } x, y \in \mathbb{Z}.
\]

(c) A set $A$ is **countable** if ...

\[
\text{it is either finite or denumerable.}
\]

(d) Two non-zero integers $a, b \in \mathbb{Z}$ are **relatively prime** if ...

\[
gcd(a, b) = 1.
\]

(e) A set $A$ is **uncountable** if ...

\[
\text{it is not countable.}
\]
22. Let $S$ be the set of all denumerable sequences of zeros and ones. Thus a typical element of $S$ is an ordered infinite-tuple such as the one illustrated below:

$$(1, 1, 0, 1, 0, 1, 0, 0, 1, \ldots) \in S.$$ 

Prove that the set $S$ is uncountable.

[Note: Your proof should not rely on knowing that some other set is uncountable.]

\textbf{Proof} First we observe that $S$ is an infinite set since, for all $n \in \mathbb{N}$, $S$ contains the element

$$e_n = (0, \ldots, 0, 1, 0, \ldots)$$

having a one in the $n$th position but zeros everywhere else.

Next, suppose by way of contradiction that $S$ is a denumerable set. Then we may list the elements of $S$ as

$$a_1 = (a_{11}, a_{12}, a_{13}, \ldots)$$
$$a_2 = (a_{21}, a_{22}, a_{23}, \ldots)$$
$$a_3 = (a_{31}, a_{32}, a_{33}, \ldots)$$
$$\vdots$$

where each $a_{ij} \in \{0, 1\}$. However, if we define the sequence

$$b = (b_1, b_2, b_3, \ldots)$$

by

$$b_n = \begin{cases} 1 & \text{if } a_{nn} = 0 \\ 0 & \text{if } a_{nn} = 1 \end{cases}$$

then $b_n \neq a_k$ for any $k \in \mathbb{N}$ which contradicts the assumption that every element of $S$ was one of the $a_k$ for some $k \in \mathbb{N}$. So $S$ is uncountable.
23. Use the principle of strong induction to prove that every integer \( n \geq 2 \) is prime or a product of primes.

[Note: You do not need to prove that the factorization of \( n \) as a product of primes is unique up to ordering of the factors.]

Proof

(Base Case) If \( n = 2 \), then \( n \) is prime, and so the theorem is true in this case.

(Induction Step) Let \( n > 2 \) and suppose the theorem is true for all integers \( k \) satisfying \( 2 \leq k < n \).

If \( n \) is prime, then there is nothing to prove.

Otherwise \( n \) is composite and we may write

\[ n = ab \]

where \( a, b \in \mathbb{N} \) and \( 2 \leq a < n \) and \( 2 \leq b < n \).

By the induction hypothesis, there exist primes \( p_1, \ldots, p_r \) and \( q_1, \ldots, q_s \) such that

\[ a = p_1 p_2 \cdots p_r \]
\[ b = q_1 q_2 \cdots q_s \]

Then

\[ n = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s \]

is a product of primes. Therefore, every integer \( n > 2 \) is the product of primes.
24. Prove that the intervals \((3, 4]\) and \([5, 8]\), which are subsets of \(\mathbb{R}\), are numerically equivalent.

[Note: Your proof should not depend on results in the textbook about cardinalities of intervals of real numbers.]

\[
\text{Proof} \quad \text{We will show that } \vert (3, 4] \vert \leq \vert [5, 8]\vert \quad \text{and} \quad \vert [5, 8]\vert \leq \vert (3, 4]\vert. \quad \text{Then by the Schroeder-Bernstein theorem, it will follow that } \vert (3, 4]\vert = \vert [5, 8]\vert.
\]

Consider the function \(f : (3, 4] \rightarrow [5, 8]\) given by \(f(x) = x + 2\). This function is injective since \(f(x_1) = f(x_2) \Rightarrow x_1 + 2 = x_2 + 2 \Rightarrow x_1 = x_2\). So \(\vert (3, 4]\vert \leq \vert [5, 8]\vert\).

On the other hand, consider the function \(g : [5, 8] \rightarrow (3, 4]\) given by \(g(x) = \frac{x}{2} + 2\). This function is injective since if \(g(x_1) = g(x_2)\) then \(\frac{x_1}{2} + 2 = \frac{x_2}{2} + 2 \Rightarrow \frac{x_1}{2} = \frac{x_2}{2} \Rightarrow x_1 = x_2\). So, \(\vert [5, 8]\vert \leq \vert (3, 4]\vert\).

Since \(\vert (3, 4]\vert \leq \vert [5, 8]\vert \) and \(\vert [5, 8]\vert \leq \vert (3, 4]\vert\),
\[
\vert (3, 4]\vert = \vert [5, 8]\vert,
\]
25. (a) Use the Euclidean algorithm to compute the greatest common divisor of 42 and 136. Show your work, and box the answer.

\[ 136 = 42 \cdot 3 + 10 \]
\[ 42 = 10 \cdot 4 + 2 \]
\[ 10 = 2 \cdot 5 + 0 \]

\[ \gcd(42, 136) = 2 \]

(b) Use the calculation from part (a) to express \( \gcd(42, 136) \) as a linear combination of 42 and 136. Show your work, and box the answer.

\[ \gcd(42, 136) = 2 \]
\[ = 42 - 10 \cdot 4 \]
\[ = 42 - (136 - 42 \cdot 3) \cdot 4 \]
\[ = 42 (13) + 136 (-4) \]