Math 290 Section 2 and 4 – Midterm Exam 3
April 1 – April 2, 2015
Professor: David Cardon, 326 TMCB, Campus Ext. 2-4863

Instructions:

• Questions 1–10 are true-false worth 3 points each. On the bubble sheet, mark A for true, B for false.

• Questions 11–20 are multiple choice worth 3 points each. Mark the correct answer on your bubble sheet.

• Questions 21–25 are written response questions worth 8 points each. Neatly write your solutions directly on the exam paper. To receive full credit you must provide complete and correct explanations.

• In written response questions words like find, show, solve, determine, or prove mean that you should give complete explanations of the reasoning involved in the finding, showing, solving, determining, or proving.

• Notes, books, and personal calculators are not allowed. A Testing Center calculator is allowed.

• No time limit.

For instructor use only:

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True-false questions

1. For nonempty sets $A$ and $B$, a relation $R$ from $A$ to $B$ is a function when every $b \in B$ appears exactly once in an ordered pair in $R$.

2. A function $f : A \to B$ is one-to-one if whenever $f(x) = f(y)$ for $x, y \in A$, we have $x = y$.

3. Every injective function is surjective.

4. There exist functions $f : A \to B$ and $g : B \to C$ such that $f$ is not surjective but $g \circ f : A \to C$ is surjective.

5. Every two denumerable sets are numerically equivalent.

6. For every set $A$, there exists a bijection from $A$ to its power set $\mathcal{P}(A)$.

7. Every infinite subset of an uncountable set is uncountable.

8. For $a, b, c, d \in \mathbb{Z}$ with $a \neq 0$ and $c \neq 0$, if $a \mid b$ and $c \mid d$, then $ac \mid (ad + bc)$.

9. For integers $a$ and $b$, not both zero, there are infinitely many integers $s$ and $t$ for which $\gcd(a, b) = as + bt$.

10. The integers 10 and 21 are relatively prime.

Multiple choice section

11. Let $A$ be a nonempty set. How many equivalence relations are there on $A$ that are also functions?
   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 4
   (f) none of the above

12. The number of functions from $A = \{x, y, z\}$ to $B = \{1, 2\}$ is
   (a) 2
   (b) 4
   (c) 8
   (d) 16
   (e) 32
   (f) none of the above.
13. The inverse of the permutation \((1 \ 2 \ 3 \ 4)
\begin{pmatrix}
3 & 2 & 4 & 1 \\
3 & 2 & 4 & 1 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
\end{pmatrix}
\) in \(S_4\) is

- (a) \((1 \ 2 \ 3 \ 4)
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
4 & 2 & 1 & 3 \\
4 & 2 & 1 & 3 \\
\end{pmatrix}
\)
- (b) \((1 \ 2 \ 3 \ 4)
\begin{pmatrix}
4 & 2 & 1 & 3 \\
4 & 3 & 2 & 1 \\
2 & 1 & 4 & 3 \\
2 & 1 & 4 & 3 \\
\end{pmatrix}
\)
- (c) \((1 \ 2 \ 3 \ 4)
\begin{pmatrix}
1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1 \\
2 & 1 & 4 & 3 \\
2 & 1 & 4 & 3 \\
\end{pmatrix}
\)
- (d) \((1 \ 2 \ 3 \ 4)
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
2 & 3 & 4 & 1 \\
2 & 3 & 4 & 1 \\
\end{pmatrix}
\)
- (e) \((1 \ 2 \ 3 \ 4)
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
2 & 3 & 4 & 1 \\
2 & 3 & 4 & 1 \\
\end{pmatrix}
\)
- (f) none of the above.

14. Which of the following sets are denumerable?

- (i) \(5\mathbb{Z}\), (ii) \(\mathbb{Q} - 2\mathbb{Z}\), (iii) \(\mathbb{R} - \mathbb{Q}\).

- (a) none of them
- (b) only (i)
- (c) only (ii)
- (d) only (iii)
- (e) only (i) and (ii)
- (f) only (i) and (iii)
- (g) only (ii) and (iii)
- (h) all of them

15. Which of the following sets have the same cardinality as that of \(\mathbb{R}\)?

- (i) \(\mathbb{R} - \{1\}\), (ii) the open interval \((-1, 5)\), (iii) the closed interval \([4, 10]\).

- (a) none of them
- (b) only (i)
- (c) only (ii)
- (d) only (iii)
- (e) only (i) and (ii)
- (f) only (i) and (iii)
- (g) only (ii) and (iii)
- (h) all of them

\[3\]
16. Any prime integer \( p > 2 \) can be written in the form \( p = 30k + r \) for integers \( k \) and \( r \) with \( 0 \leq r < 30 \) when the value of \( r \) is
(a) 0
(b) 1
(c) 2
(d) 6
(e) 15
(f) none of the above.

17. Which of the following prime numbers appear in the canonical factorization of 90?
(a) 2
(b) 5
(c) 7
(d) all of them
(e) some, but not all of them
(f) none of them

18. The smallest prime factor of 529 is
(a) 3
(b) 7
(c) 11
(d) 19
(e) 23
(f) none of the above.

19. Which of the following sets are denumerable?
(i) \( \{ x \in \mathbb{N} : x \text{ is prime} \} \),
(ii) \( \{ y \in \mathbb{N} : 15 \mid y \} \),
(iii) \( \{ z \in \mathbb{N} : \gcd(z, 6) = 1 \} \).
(a) none of them
(b) only (i)
(c) only (ii)
(d) only (iii)
(e) only (i) and (ii)
(f) only (i) and (iii)
(g) only (ii) and (iii)
(h) all of them
20. For the sets,

\[ A = \{ n \in \mathbb{Z} : n \text{ is composite} \}, \quad B = 2^{\mathbb{N}}, \quad C = A \times B, \]

which of the following hold?

(a) \( |A| < |B| \) and \( |B| \leq |C| \)
(b) \( |B| \leq |C| \) and \( |C| \leq |A| \)
(c) \( |C| < |A| \) and \( |A| < |B| \)
(d) \( |A| = |C| \) and \( |C| < |B| \)
(e) \( |C| < |B| \) and \( |B| < |A| \)
(f) none of the above

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**Written Answer Section**

21. Let \( a, b, c, n \in \mathbb{Z} \) with \( n \geq 2 \). Prove that if \( ac \equiv bc \pmod{n} \) and \( \gcd(c, n) = 1 \), then \( a \equiv b \pmod{n} \).
22. Let \( f : \mathbb{Z}_5 \rightarrow \mathbb{Z}_5 \) be defined by \( f([a]) = [2a + 1] \).

(a) Prove that \( f \) is well-defined.

(b) Prove that \( f \) is a bijection.
23. Prove that the function $f: \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{3\}$ defined by

$$f(x) = \frac{3x}{x - 1}$$

is a bijection, and determine its inverse $f^{-1}$. 
24. Let $A$ and $B$ be the subsets of $\mathbb{R}$ given by

\[ A = [1, 4] \cup \{5\} \quad \text{and} \quad B = (2, 5). \]

Prove that $|A| = |B|$.

(Hint: Schröder-Bernstein)
25. (a) Use the Euclidean algorithm to compute \( \gcd(182, 308) \). Show your work, and box the answer.

(b) Use reverse substitution in calculation from part (a) to find integers \( x \) and \( y \) such that \( \gcd(182, 308) = 182x + 308y \). Show your work, and box the answer.