Midterm exam 2 is in the Testing Center on Monday, Mar 1 and Tuesday, Mar 2. There is no late day.

The exam will emphasize the topics in Chapters 4, 5, 6, and 8. (Sections 8.5 and 8.6 will not be on this exam.) The exam questions will be based on textbook examples and homework problems.

Sample Questions and Preparation Suggestions

1. (Exercise 4.7) Prove that $3 \mid (2n^2 + 1)$ if and only if $3 \nmid n$.

2. Let $a, b, n \in \mathbb{Z}$, where $n \geq 2$. By directly applying the definition of congruence (rather than by quoting a result about congruences), show that if $a \equiv 7 \pmod{n}$ and $b \equiv 3 \pmod{n}$, then $ab \equiv 21 \pmod{n}$.

3. (Exercise 4.20) Let $x \in \mathbb{R}$. Prove that if $3x^4 + 1 \leq x^7 + x^3$, then $x > 0$.

4. (Result 4.18) Prove that, for every two sets $A$ and $B$, $A - B = A \cap \overline{B}$.

5. (Theorem 4.21 4a) If $A$ and $B$ are sets, prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

6. (Exercise 4.45) Let $A$, $B$, and $C$ be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

7. (Example 5.5) Disprove the statement: “Let $n \in \mathbb{Z}$. If $n^2 + 3n$ is even, then $n$ is odd”.

8. Figure out how to make true-false questions about proof by contradiction from the discussion on page 111. Also consider how to make true-false questions or multiple choice questions based on Figure 5.2 on page 117.

9. (Theorem 5.16) Prove that the real number $\sqrt{2}$ is irrational.

10. What is an “existence theorem” or an “existence proof”? How can you disprove an existence statement?

11. (Exercise 5.20) If $x$ and $y$ are positive real numbers, prove that $\sqrt{x} + \sqrt{y} \neq \sqrt{x + y}$.

12. Result 5.22, Result 5.23, and Exercise 5.33, which all use the Intermediate Value Theorem from Calculus.

13. Consider how you might write a multiple choice question based on the proof evaluation exercises 5.43, 5.44, 5.45 and 5.46.


15. Use mathematical induction to prove that

\[
\begin{align*}
(a) \quad \sum_{k=1}^{n} k & = \frac{n(n+1)}{2} \quad \text{for } n \in \mathbb{N}. \\
(b) \quad \sum_{k=1}^{n} k^2 & = \frac{n(n+1)(2n+1)}{6} \quad \text{for } n \in \mathbb{N}.
\end{align*}
\]
(c) \[ 2 + 7 + 12 + 17 + \cdots + (5n + 2) = \frac{(n+1)(5n+4)}{2} \] for nonnegative integers \( n \).

16. (Result 6.10) For every integer \( n \geq 5 \), prove that \( 2^n > n^2 \).

17. (Exercise 6.11) Prove that \( \frac{1}{3^4} + \frac{1}{4^5} + \cdots + \frac{1}{(n+2)(n+3)} = \frac{n}{3n+9} \) for every positive integer \( n \).

18. (Exercise 6.14) Prove \( n! > 2^n \) for every integer \( n \geq 4 \).

19. (Exercise 6.17 – Bernoulli’s Identity) For every real number \( x > -1 \) and every positive integer \( n \), prove that \( (1 + x)^n \geq 1 + nx \).

20. Use the method of minimum counterexample to prove that \( 6 \mid n(n^2 - 1) \) for every positive integer \( n \).

21. Explain how the strong principle of induction and the principle of induction are different. Find an example in the book that uses strong induction and make sure you understand how the ‘strong’ induction hypothesis was used in the argument where ‘regular’ induction would have been inadequate.

22. Think of ways to create multiple choice questions based on the proof evaluation exercises 6.48, 6.49, 6.50.

23. Understand relations and the reflexive property, the symmetric property, and the transitive property. Be able to give examples of relations on a set like \( A = \{1, 2, 3, 4\} \) that satisfy none, one, two, or all three of those properties. This would be a good candidate for a multiple choice question.

24. Understand equivalence relations and equivalence classes.

25. Theorem 8.2, 8.3, and 8.4 are so important that you should memorize their proofs. There will be a question on the test about one of these theorem.

This is not useless memorization. The ideas in the proofs are relatively simple but extremely important and are foundational for almost all of mathematics.

26. Let \( A = \{(a, b) : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\} \).

Let \( R \) be the relation on \( A \) defined by the rule

\[ (a, b) \sim (c, d) \iff ad = bc. \]

where in this exercise we are using the more elegant notation \((a, b) \sim (c, d)\) to mean exactly the same thing as the uglier notation \((a, b)R(c, d)\).

(a) Prove that the relation \( R \) is reflexive: If \((a, b) \in A\), then \((a, b) \sim (a, b)\).

(b) Prove that the relation \( R \) is symmetric: If \((a, b) \in A\), \((c, d) \in A\), and \((a, b) \sim (c, d)\), then \((c, d) \sim (a, b)\).

(c) Prove that the relation \( R \) is transitive: If \((a, b) \in A\), \((c, d) \in A\), and \((e, f) \in A\) and if \((a, b) \sim (c, d)\) and \((c, d) \sim (e, f)\), then \((a, b) \sim (e, f)\).

Do the equivalence classes in this example seem familiar?

27. Questions about relations and equivalence relations like exercises 8.1–8.27 could be on the exam.