1. Let $P = \{A_\alpha : \alpha \in I\}$ be a partition of a nonempty set $A$. Then there exists an equivalence relation $R$ on $A$ such that $P$ is the set of equivalence classes determined by $R$.

(a) True
(b) False

2. In $\mathbb{Z}_{12}$, if $[a] \cdot [b] = [0]$, then it follows that $[a] = [0]$ or $[b] = [0]$.

(a) True
(b) False

3. There exist functions $f : A \to B$ and $g : B \to C$, such that $f$ is not surjective, but $g \circ f : A \to C$ is surjective.

(a) True
(b) False

4. Let $f : A \to B$ and $g : B \to A$ be functions such that $g \circ f = i_A$, where $i_A$ is the identity function on $A$. Then $f$ is injective and $g$ is surjective.

(a) True
(b) False

5. Let $f : A \to B$ and $g : B \to A$ be functions such that $g \circ f = i_A$, where $i_A$ is the identity function on $A$. Then $g$ is necessarily injective.

(a) True
(b) False

6. The mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x + 2$ is a bijection.

(a) True
(b) False

7. The mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$ is a bijection.

(a) True
(b) False
8. If \( A, B, \) and \( C \) are nonempty sets such that \( A \subset B \subset C \), then \( |A| < |B| < |C| \).
   (a) True
   (b) \[ \text{False} \]

9. Every uncountable set contains a denumerable subset.
   (a) True
   (b) False

10. A set \( A \) is denumerable if and only if there exists an injective function \( f : \mathbb{N} \to A \).
    (a) True
    (b) False

11. In \( \mathbb{Z}_8, [-13] : [138] = \)
    (a) [0]  (b) [1]  (c) [2]  (d) [3]  (e) [4]  (f) [5]  (g) [6]  (h) [7]

12. Evaluate the proposed proof of the following result:
    **Result:** The sets \((0, \infty)\) and \([0, \infty)\) are numerically equivalent
    
    **Proof.** Define the function \( f : (0, \infty) \to [0, \infty) \) by \( f(x) = x \).
    
    First we show that \( f \) is one-to-one. Let \( a, b \in (0, \infty) \) and assume that \( f(a) = f(b) \). Then \( a = b \) and so \( f \) is one-to-one.
    
    Next, we show that \( f \) is onto. Let \( r \in [0, \infty) \). Since \( f(r) = r \), the function \( f \) is onto.
    
    Since \( f \) is bijective, \(|(0, \infty)| = |[0, \infty)|\).
    
    Choose the most accurate response.
    (a) The proof is correct.
    (b) The proof correctly shows that \( f \) is one-to-one, but the proof that \( f \) is onto has a flaw.
    (c) The proof correctly shows that \( f \) is onto, but the proof that \( f \) is one-to-one has a flaw.
    (d) The argument that \( f \) is one-to-one has a flaw, and the argument that \( f \) is onto also has a flaw.

13. Which of the following functions \( f : \mathbb{Z}_{10} \to \mathbb{Z}_{10} \) is injective?
    (a) \( f([a]) = [5a + 1] \)
    (b) \( f([a]) = [6a + 3] \)
    (c) \( f([a]) = [3a + 2] \)
    (d) \( f([a]) = [2a + 7] \)
    (e) None of the above.
    (f) All of the above.
14. Let \( A = \{1, 2, 3\} \) and \( B = \{a, b, c\} \). If \( f : A \to B \) is a function, which of the following possibly might represent the relation \( f^{-1} \)?

(a) \( \{(a, 1), (b, 1), (c, 1)\} \)
(b) \( \{(1, a), (2, c), (3, b)\} \)
(c) \( \{(c, 1), (b, 2), (a, 1)\} \)
(d) \( \{(b, 2), (b, 3), (a, 1)\} \)
(e) \( \{(1, a), (1, b), (1, c)\} \)
(f) None of the above

15. How many equivalence relations are there on the set \( A = \{1, 2, 3\} \)?

(a) 1  (b) 2  (c) 3  (d) 4  (e) 5  (f) 6  (g) 8  (h) 256

16. How many functions are there from \( \mathbb{Z}_4 \) to \( \mathbb{Z}_5 \)?

(a) 4  (b) 5  (c) 4! = 24  (d) 5! = 120  (e) 4^5 = 1024  (f) 5^4 = 625  (g) none of these

17. Which of the following functions would be most useful for proving that \( |[0, 1)| = |[1, \infty)| \)?

(a) \( f(x) = \frac{1}{x} \)
(b) \( f(x) = \frac{1+x}{x} \)
(c) \( f(x) = \frac{x}{1+x} \)
(d) \( f(x) = \frac{1-x}{1+x} \)
(e) \( f(x) = \frac{x}{x-1} \)
(f) \( f(x) = \frac{x+1}{x-1} \)
(g) \( f(x) = \frac{1}{1-x} \)
(h) None of the given functions would be useful.

18. Which of the following statements is true?

(a) If \( A \) is denumerable, then \(|A| = |\mathbb{R}|\).
(b) There exists a surjective function \( f : \mathbb{Q} \to \mathbb{R} \).
(c) If \( A \) is uncountable, then \(|A| = |\mathbb{R}|\).
(d) If \( A, B, \) and \( C \) are sets with \( A \subseteq B \subseteq C \) such that \( A \) and \( C \) are countable, then \( B \) is countable.
(e) If \( A \) is denumerable and \( A \) is a proper subset of \( B \), then \( B \) is uncountable.
(f) None of the above is true.
19. Let \( \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 1 & 3 & 6 & 5 \end{pmatrix} \) and \( \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 6 & 2 & 5 \end{pmatrix} \). Which of the following is \( \beta \circ \alpha^{-1} \)?

(a) \( \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 6 & 3 & 5 & 2 \end{pmatrix} \)

(b) \( \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 5 & 2 & 6 \end{pmatrix} \)

(c) \( \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 2 & 5 & 3 \end{pmatrix} \)

(d) \( \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 1 & 4 & 6 \end{pmatrix} \)

(e) None of the above.

20. Which of the following sets has cardinality different from that of the others?

(a) \( \mathbb{R} \)

(b) the open interval \((0, 1)\)

(c) the open interval \((0, 2)\)

(d) the power set of \( \mathbb{N} \)

(e) the power set of \( \mathbb{Q} \)

(f) the power set of \( \mathbb{R} \)

(g) All of the sets given above have the same cardinality.
Instructions: Neatly write the solutions directly on the exam paper. Complete explanations are required for full credit.

21. Prove that the function \( f : \mathbb{Z}_{13} \to \mathbb{Z}_{13} \) defined by the formula

\[
f([a]) = [a^2 + a + 2]
\]

is well-defined.

Note 1: This problem very similar to Result 9.6 on page 205. The answer is also very similar to the proofs of the facts that addition and multiplication are well-defined in \( \mathbb{Z}_n \).

Solution. If \([a] = [b] \in \mathbb{Z}_{13}\), we must show that \( f([a]) = f([b]) \). In other words, we must show that

\[
[a^2 + a + 2] = [b^2 + b + 2].
\]

So, let \([a] = [b]\), where \([a], [b] \in \mathbb{Z}_{13}\). Then \(a \equiv b \pmod{13}\) and

\[
a - b = 13m
\]

for some \(m \in \mathbb{Z}\). Then

\[
(a^2 + a + 2) - (b^2 + b + 2) = a^2 - b^2 + a - b
\]

\[
= (a - b)(a + b) + (a - b)
\]

\[
= (a - b)(a + b + 1)
\]

\[
= 13m(a + b + 1)
\]

Since, \(m(a + b + 1)\) is an integer, \(13 \mid ((a^2 + a + 2) - (b^2 + b + 2))\) and

\[
a^2 + a + 2 \equiv b^2 + b + 2 \pmod{13}.
\]

Hence, \([a^2 + a + 2] = [b^2 + b + 2]\). This shows that \( f \) is well-defined.

Note 2: The values of \( f \) are

\[
\begin{align*}
f([0]) &= [2], f([1]) = [4], f([2]) = [8], f([3]) = [1], f([4]) = [9], f([5]) = [6], f([6]) = [5], \
f([7]) &= [6], f([8]) = [9], f([9]) = [1], f([10]) = [8], f([11]) = [4], f([12]) = [2].
\end{align*}
\]

Calculating this table of values does not show that the attempt to define a function with the formula \( f([a]) = [a^2 + a + 2] \) actually defines a function.
22. Let $S$ be the set of all sequences of zeros and ones. Thus a typical element of $S$ is an ordered infinite-tuple such as the one illustrated below:

$$(1, 1, 0, 1, 0, 1, 0, 0, 1, \ldots) \in S.$$ 

Prove that $S$ is uncountable.

**Note:** This problem is very similar to the problem of showing that the interval $(0, 1)$ is uncountable.

**Proof.** Suppose, by way of contradiction, that $S$ is countable. Since $S$ is an infinite set, $S$ must be denumerable. Therefore, $S$ may be expressed as $S = \{a_1, a_2, a_3, \ldots\}$. For each $i \in \mathbb{N}$, we may write the sequence $a_k$ as

$$a_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}, \ldots)$$

where $a_{ij}$ is the $j$th term of the sequence $a_i$ and $s_{ij}$ is either 0 or 1. Now let $b$ be the sequence

$$b = (b_1, b_2, b_3, b_4, \ldots)$$

defined by

$$b_n = \begin{cases} 
1 & \text{if } a_{nn} = 0, \\
0 & \text{if } a_{nn} = 1.
\end{cases}$$

Since $b$ is a sequences of zeros and ones, $b \in S$. However, since $b_n \neq s_{nn}$, $b \neq s_n$ for any $n \in \mathbb{N}$. Since $S = \{a_1, a_2, a_3, \ldots\}$, this would imply $b \notin S$. So, $b \in S$ and $b \notin S$, which is a contradiction. Therefore, $S$ is uncountable. \qed
23. Let $A$ and $B$ be disjoint denumerable sets. Show that $A \times B$ is denumerable.

**Note:** This is Result 10.5 in the text. The book has prettier pictures than this solution.

**Proof.** Since $A$ and $B$ are denumerable, we may write $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$. Then create the semi-infinite array such that the pair $(a_i, b_j)$ is in the $i$th row and $j$th column. Then every element of $A \times B$ appears in the array exactly one time.

\[
\begin{array}{cccc}
(a_1, b_1) & (a_1, b_2) & (a_1, b_3) & (a_1, b_4) \\
(a_2, b_1) & (a_2, b_2) & (a_2, b_3) & (a_2, b_4) \\
(a_3, b_1) & (a_3, b_2) & (a_3, b_3) & (a_3, b_4) \\
(a_4, b_1) & (a_4, b_2) & (a_4, b_3) & (a_4, b_4) \\
& & & \\
\vdots & & & \ddots
\end{array}
\]

Now we define a function $f : \mathbb{N} \rightarrow A \times B$ by moving through the elements of the array in a particular order. Begin with the element $(a_1, b_1)$. Then move to the second entry of the top row and move diagonally downward to the left until reaching the first column, then move to the third element of the first row and move downward and to the left until reach the first column, then move to the fourth entry of the first row and move downward and to the left until reaching the first column, and so forth. The first few values of the function $f$ are given below:

\[
\begin{align*}
 f(1) &= (a_1, b_1) \quad \text{(subscripts add to 2)} \\
 f(2) &= (a_1, b_2) \quad \text{(subscripts add to 3)} \\
 f(3) &= (a_2, b_1) \\
 f(4) &= (a_1, b_3) \quad \text{(subscripts add to 4)} \\
 f(5) &= (a_2, b_2) \\
 f(6) &= (a_1, b_3) \\
 f(7) &= (a_4, b_1) \quad \text{(subscripts add to 5)} \\
 & \quad \vdots
\end{align*}
\]

This procedure create a path through the array in which each element is counted exactly one time which make the function $f : \mathbb{N} \rightarrow A \times B$ a bijection. Thus $A \times B$ is denumerable. \qed
24. Let $A$, $B$, and $C$ be nonempty sets and suppose $f: A \to B$ and $g: B \to C$ are functions.

(a) If $f$ and $g$ are both injective, show that $g \circ f: A \to C$ is injective.

Proof. Let $a, b \in A$ and assume $(g \circ f)(a) = (g \circ f)(b)$. Then $g(f(a)) = g(f(b))$. Since $g$ is injective, $f(a) = f(b)$. Since $f$ is injective $a = b$. This shows $g \circ f$ is injective. \qed

(b) If $f$ and $g$ are both surjective, show that $g \circ f: A \to C$ is surjective.

Proof. Assume $f$ and $g$ are both surjective. Let $c \in C$. Since $g$ is surjective, there exists $b \in B$ such that $g(b) = c$. Since $f$ is surjective, there exists $a \in A$ such that $f(a) = b$. Then $g(f(a)) = g(b) = c$.

This shows that $g \circ f$ is surjective. \qed
25. Let $R$ be the relation defined on $\mathbb{Z}$ by $a R b$ if $2a + b \equiv 0 \pmod{3}$.

(a) Show that $R$ is reflexive.

Proof. Let $n \in \mathbb{Z}$. Since $2n + n = 3n \equiv 0 \pmod{3}$, it follows that $n R n$. Thus $R$ is reflexive. \qed

(b) Show that $R$ is symmetric.

Proof. Assume $a R b$. Then

\[
2a + b \equiv 0 \pmod{3} \\
2a + b = 3m \quad \text{for some } m \in \mathbb{Z} \\
-a - 2b = 3m - 3a - 3b \quad \text{(Subtract } 3a + 3b \text{ from both sides.)} \\
a + 2b = 3(-m + a + b)
\]

This shows that $a + 2b \equiv 0 \pmod{3}$. In other words $b R a$. So, $R$ is reflexive. \qed

(c) Show that $R$ is transitive.

Proof. Assume $a R b$ and $b R c$. Then $a + 2b \equiv 0 \pmod{3}$ and $b + 2c \equiv 0 \pmod{3}$. Thus,

\[
a + 2b = 3m \quad \text{and} \quad b + 2c = n
\]

for some $m, n \in \mathbb{Z}$. Adding these two equations gives

\[
(a + 2b) + (b + 2c) = 3m + 3n \\
a + 2c = 3m + 3n - 3b = 3(m + n - b) \\
a + 2c \equiv 0 \pmod{3}
\]

So, $a R c$. This proves that $R$ is transitive. \qed