Math 290 Midterm Exam 3  
Sections 2 and 4 — Winter 2010  
Wed, Mar 24 through Thu, Mar 25 – Testing Center  
Professor: David Cardon, 302 TMCB, Campus Ext. 2-4863  

Instructions:

- There are 25 questions for a total of 100 points.

- Questions 1–20 are true-false and multiple choice questions. Each is worth 3 points. Mark the answers on the bubble sheet.

- Questions 21–25 are essay questions. Each is worth 8 points. Neatly write the answers to the essay questions directly on the exam paper. If a solution requires more space than given, you may continue on the back of the page.

- For the essay questions, you are expected to give complete and clear explanations of the reasoning involved. A final answer without explanation is usually inadequate, except where it is specifically stated that no explanation is necessary.

- Notes, books, and calculators are not allowed.

- No time limit.
True-False and Multiple Choice Section: Questions 1–20

Instructions: On the bubble sheet mark the best answer.

1. Let \( P = \{ A_\alpha : \alpha \in I \} \) be a partition of a nonempty set \( A \). Then there exists an equivalence relation \( R \) on \( A \) such that \( P \) is the set of equivalence classes determined by \( R \).

   (a) True
   (b) False

2. In \( \mathbb{Z}_{12} \), if \([a] \cdot [b] = [0]\), then it follows that \([a] = [0]\) or \([b] = [0]\).

   (a) True
   (b) False

3. There exist functions \( f : A \to B \) and \( g : B \to C \), such that \( f \) is not surjective, but \( g \circ f : A \to C \) is surjective.

   (a) True
   (b) False

4. Let \( f : A \to B \) and \( g : B \to A \) be functions such that \( g \circ f = i_A \), where \( i_A \) is the identity function on \( A \). Then \( f \) is injective and \( g \) is surjective.

   (a) True
   (b) False

5. Let \( f : A \to B \) and \( g : B \to A \) be functions such that \( g \circ f = i_A \), where \( i_A \) is the identity function on \( A \). Then \( g \) is necessarily injective.

   (a) True
   (b) False

6. The mapping \( f : \mathbb{Z} \to \mathbb{Z} \) defined by \( f(x) = 3x + 2 \) is a bijection.

   (a) True
   (b) False

7. The mapping \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 3x + 2 \) is a bijection.

   (a) True
   (b) False
8. If $A$, $B$, and $C$ are nonempty sets such that $A \subset B \subset C$, then $|A| < |B| < |C|$.

(a) True
(b) False

9. Every uncountable set contains a denumerable subset.

(a) True
(b) False

10. A set $A$ is denumerable if and only if there exists an injective function $f : \mathbb{N} \to A$.

(a) True
(b) False

11. In $\mathbb{Z}_8$, $[-13] \cdot [138] =$

(a) [0]  (b) [1]  (c) [2]  (d) [3]  (e) [4]  (f) [5]  (g) [6]  (h) [7]

12. Evaluate the proposed proof of the following result:

**Result:** The sets $(0, \infty)$ and $[0, \infty)$ are numerically equivalent

**Proof.** Define the function $f : (0, \infty) \to [0, \infty)$ by $f(x) = x$.

First we show that $f$ is one-to-one. Let $a, b \in (0, \infty)$ and assume that $f(a) = f(b)$. Then $a = b$ and so $f$ is one-to-one.

Next, we show that $f$ is onto. Let $r \in [0, \infty)$. Since $f(r) = r$, the function $f$ is onto.

Since $f$ is bijective, $|(0, \infty)| = |[0, \infty)|$.

Choose the most accurate response.

(a) The proof is correct.
(b) The proof correctly shows that $f$ is one-to-one, but the proof that $f$ is onto has a flaw.
(c) The proof correctly shows that $f$ is onto, but the proof that $f$ is one-to-one has a flaw.
(d) The argument that $f$ is one-to-one has a flaw, and the argument that $f$ is onto also has a flaw.

13. Which of the following functions $f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ is injective?

(a) $f([a]) = [5a + 1]$
(b) $f([a]) = [6a + 3]$
(c) $f([a]) = [3a + 2]$
(d) $f([a]) = [2a + 7]$
(e) None of the above.
(f) All of the above.
14. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. If $f: A \rightarrow B$ is a function, which of the following possibly might represent the relation $f^{-1}$?

(a) $\{(a, 1), (b, 1), (c, 1)\}$
(b) $\{(1, a), (2, c), (3, b)\}$
(c) $\{(c, 1), (b, 2), (a, 1)\}$
(d) $\{(b, 2), (b, 3), (a, 1)\}$
(e) $\{(1, a), (1, b), (1, c)\}$
(f) None of the above

15. How many equivalence relations are there on the set $A = \{1, 2, 3\}$?

(a) 1  (b) 2  (c) 3  (d) 4  (e) 5  (f) 6  (g) 8  (h) 256

16. How many functions are there from $\mathbb{Z}_4$ to $\mathbb{Z}_5$?

(a) 4  (b) 5  (c) $4! = 24$  (d) $5! = 120$  (e) $4^5 = 1024$  (f) $5^4 = 625$  (g) none of these

17. Which of the following functions would be most useful for proving that $|[0, 1)| = |[1, \infty)|$?

(a) $f(x) = \frac{1}{x}$
(b) $f(x) = \frac{1+x}{x}$
(c) $f(x) = \frac{1}{1+x}$
(d) $f(x) = \frac{1-x}{1+x}$
(e) $f(x) = \frac{x}{x-1}$
(f) $f(x) = \frac{x+1}{x-1}$
(g) $f(x) = \frac{1}{1-x}$
(h) None of the given functions would be useful.

18. Which of the following statements is true?

(a) If $A$ is denumerable, then $|A| = |\mathbb{R}|$.
(b) There exists a surjective function $f: \mathbb{Q} \rightarrow \mathbb{R}$.
(c) If $A$ is uncountable, then $|A| = |\mathbb{R}|$.
(d) If $A$, $B$, and $C$ are sets with $A \subseteq B \subseteq C$ such that $A$ and $C$ are countable, then $B$ is countable.
(e) If $A$ is denumerable and $A$ is a proper subset of $B$, then $B$ is uncountable.
(f) None of the above is true.
19. Let \( \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 1 & 3 & 6 & 5 \end{pmatrix} \) and \( \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 6 & 2 & 5 \end{pmatrix} \). Which of the following is \( \beta \circ \alpha^{-1} \)?

(a) \( \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 6 & 3 & 5 & 2 \end{pmatrix} \)

(b) \( \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 5 & 2 & 6 \end{pmatrix} \)

(c) \( \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 2 & 5 & 3 \end{pmatrix} \)

(d) \( \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 1 & 4 & 6 \end{pmatrix} \)

(e) None of the above.

20. Which of the following sets has cardinality different from that of the others?

(a) \( \mathbb{R} \)

(b) the open interval \((0, 1)\)

(c) the open interval \((0, 2)\)

(d) the power set of \(\mathbb{N}\)

(e) the power set of \(\mathbb{Q}\)

(f) the power set of \(\mathbb{R}\)

(e) the set \(2^\mathbb{N}\)

(g) All of the sets given above have the same cardinality.
Essay Section: Questions 21–25

Instructions: Neatly write the solutions directly on the exam paper. Complete explanations are required for full credit.

21. Prove that the function \( f : \mathbb{Z}_{13} \to \mathbb{Z}_{13} \) defined by the formula

\[
 f([a]) = [a^2 + a + 2]
\]

is well-defined.
22. Let $S$ be the set of all sequences of zeros and ones. Thus a typical element of $S$ is an ordered
infinite-tuple such as the one illustrated below:

$$(1, 1, 0, 1, 0, 1, 0, 0, 1, \ldots) \in S.$$ 

Prove that $S$ is uncountable.
23. Let $A$ and $B$ be disjoint denumerable sets. Show that $A \times B$ is denumerable.
24. Let $A$, $B$, and $C$ be nonempty sets and suppose $f: A \to B$ and $g: B \to C$ are functions.

(a) If $f$ and $g$ are both injective, show that $g \circ f: A \to C$ is injective.

(b) If $f$ and $g$ are both surjective, show that $g \circ f: A \to C$ is surjective.
25. Let $R$ be the relation defined on $\mathbb{Z}$ by $a R b$ if $2a + b \equiv 0 \pmod{3}$.

(a) Show that $R$ is reflexive.

(b) Show that $R$ is symmetric.

(c) Show that $R$ is transitive.