Math 290
Sample Exam 3

Note that the first 10 questions are true-false. Mark A for true, B for false. Questions 11 through 20 are multiple choice—mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and correctly.

True-false questions
1. Let \( A \) be an uncountable set. There is no uncountable set \(|B|\) with \(|B| < |A|\).
2. A set \( A \) is countable if and only if there is a bijection \( f : \mathbb{N} \to A \).
3. Every subset of an uncountable set is either finite or denumerable.
4. For every nonempty set \( A \) the sets \( \mathcal{P}(A) \) and \( \{0,1\}^A \) are numerically equivalent.
5. Let \( S = \{(a,b) \in \mathbb{N} \times \mathbb{N} : a \leq b^2\} \). Then \( S \) is denumerable.
6. If \( A \) and \( B \) are nonempty sets, then \(|A \times B| \leq |A|\).
7. Let \( p \in \mathbb{Z} \) with \( p \geq 2 \). Then \( p \) is prime if and only if for all \( a \in \mathbb{Z} \), either \( p|a \) or \( (a,p) = 1 \).
8. Suppose that \( n, d, q, r \in \mathbb{Z} \) with \( n, d \neq 0 \) and \( n = qd + r \). Then gcd\((n,d) = \gcd(d,r)\).
9. Let \( a, b, c \in \mathbb{Z} \) with \( a, b \neq 0 \). If \( a|c \) and \( b|c \) then \( ab|c \).
10. The gcd of 7081 and 4453 is between 50 and 75.

Multiple choice section
On all problems, choose the most complete correct answer.

11. The set \( \mathcal{P}(\mathbb{R}) \) has cardinality which is
   a) less than \( |\mathbb{R}| \)  b) equal to \( |\mathbb{R}| \)  c) bigger than \( |\mathbb{R}| \)  d) all of the above  e) none of the above

12. Evaluate the proof of the given theorem:
   \textbf{Theorem:} Let \( a, b, c \in \mathbb{Z} \) be nonzero. If \( a|bc \) and \( a \nmid b \) then \( a|c \).
   \textbf{Proof:} Assume \( a|bc \). This implies \( bc = ax \) for some \( x \in \mathbb{Z} \). Since \( a \) doesn’t divide \( b \), we must have \( a \) divides \( c \).
   a) The theorem and proof are correct.
   b) The theorem is correct, but the proof makes an unwarranted implication.
   c) The theorem is incorrect, but the proof is correct and proves something else.
   d) The theorem is false, and the proof makes an unjustified step.
   e) all of the above
   f) none of the above

13. Which of the following sets has the same cardinality as \( \mathbb{N} \).
   a) \( \mathbb{N} \times \mathbb{N} \)
   b) \( \mathbb{Z} \times \mathbb{Q} \)
   c) \( \mathbb{Q} \)
   d) \( \mathbb{Q}^+ \)
   e) all of the above
   f) none of the above
14. Which of the following sets has cardinality different than the others:
a) $\mathbb{R}$  
b) the open interval $(0, 1)$  
c) the closed set $[0, 1]$  
d) the positive irrational numbers  
e) The power set of $\mathbb{N}$  
f) None of the above.

15. Which of the following statements is true?
a) If $A$ is an uncountable set then $|A| = |\mathbb{R}|$.
b) There exists an injective function $f : \mathbb{R} \to \mathbb{Q}$.
c) Every infinite subset of $\mathbb{R}$ is denumerable.
d) If $A$ and $B$ are two sets such that $A$ is denumerable and $|A| < |B|$ then $B$ is uncountable.  
e) If $A$, $B$, and $C$ are sets with $A \subseteq B \subseteq C$, and $A$ and $B$ are countable, then $C$ is countable.  
f) None of the above.

16. Which of the following functions would be most useful to prove that $|[0, 4]| = |[6, 10]|$?
a) $f(x) = x + 5$  
b) $f(x) = (x - 2)^2 + 6$  
c) $f(x) = 10 - x$  
d) $f(x) = -\frac{1}{2}x^2 + 3x + 6$
17. Which of (a) through (d) is \textbf{not} a true fact about the gcd of two nonzero integers $a$ and $b$?

\begin{itemize}
  \item a) The gcd divides both $a$ and $b$.
  \item b) The gcd can be written as a linear combination of $a$ and $b$.
  \item c) The gcd is smaller than both $a$ and $b$.
  \item d) Let $c \in \mathbb{Z}$, $c \neq 0$. If $c|a$ and $c|b$ then $c$ divides the gcd.
  \item e) None of the above-all of these facts are true.
\end{itemize}

18. Suppose that $a, b, c \in \mathbb{Z}$ with $a \neq 0$. Suppose $a|bc$. Which of the following must be true?

\begin{itemize}
  \item a) If $a \nmid b$ then $a|c$.
  \item b) If $a \nmid c$ then $\gcd(a, b) \neq 1$.
  \item c) If $a \neq b$ then $a|c$.
  \item d) If $a$ and $b$ are even then $c$ is odd.
  \item e) None of the above-all of these facts are false.
\end{itemize}

19. Let $n$ be an odd integer greater than 1. Suppose that for some $m \in \mathbb{Z}$, $n|(7m + 5)$ and $n|(35m + 31)$. Then the value of $n$ could be

\begin{itemize}
  \item a) 1
  \item b) 2
  \item c) 3
  \item d) 5
  \item e) 6
  \item f) 8
  \item g) 32
  \item h) None of the above.
\end{itemize}

20. How many primes are there that are one less than a square (i.e. $p = k^2 - 1$ for some $k \in \mathbb{N}$)?

\begin{itemize}
  \item a) 0
  \item b) 1
  \item c) 2
  \item d) 3
  \item e) 4
  \item f) Infinitely many
  \item g) None of the above.
\end{itemize}
Essay Section

21. State and prove Euclid’s lemma.

22. Let $A$ be a denumerable set, and let $B = \{x, y\}$. Prove that $A \times B$ is denumerable.

23. Compute $\gcd(2651, 1687)$, and write it in the form $2651x + 1687y$ for some $x, y \in \mathbb{Z}$.

24. Define the boldface terms by completing the sentences.
   A set $A$ is **uncountable** if
   A **prime number** is
   A set $A$ is **denumerable** if
   Two numbers $a, b \in \mathbb{Z}$ are **relatively prime** if
   Two sets $A$ and $B$ are **numerically equivalent** if

25. Prove that the open interval $(0, 1)$ of real numbers is uncountable. The proof should not refer to other uncountable sets.