In this section, we state the Schröder-Berstein Theorem and consider several examples. In the following section, we prove the theorem.

Recall that we write $|A| \leq |B|$ if and only if there exists an injective function from $A$ to $B$. We write $|A| = |B|$ if and only if there exists a bijection from $A$ to $B$.

**Schröder-Bernstein Theorem.** Let $A$ and $B$ be nonempty sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

**Example 1.** Use the Schröder-Berstein Theorem to show that the open interval $(3, 6)$ and the closed interval $[4, 8]$ are numerically equivalent.

**Solution.** The linear function $f : (3, 6) \to [4, 8]$ defined by $f(x) = x + 1$ is easily seen to be injective. The linear function $g : [4, 8] \to (3, 6)$ defined by $g(x) = \frac{x - 3}{2}$ is also easily seen to be injective. (As an exercise, verify for yourself that $f$ and $g$ are injective.) So, the Schröder-Berstein Theorem implies that there exists a bijection from $(3, 6)$ to $[4, 8]$. In other words, $|(3, 6)| = |[4, 8]|$. □

**Example 2.** Use the Schröder-Berstein Theorem to show that $|(0, 1)| = |(0, 1]|$.

**Solution.** Since $(0, 1) \subset (0, 1]$, we have $|(0, 1)| \leq |(0, 1]|$. On the other hand, the function $f : (0, 1] \to (0, 1)$ defined by $f(x) = x/2$ is injective. So, $|(0, 1]| \leq |(0, 1)|$. By the Schröder-Bernstein Theorem, $|(0, 1)| = |(0, 1]|$. □

**Example 3.** Prove that $|(0, 1)| = |(0, 1]|$ by finding an explicit bijection $h : (0, 1] \to (0, 1)$.

**Solution.** The main idea in this example is similar to the main idea in the proof of the Schröder-Berstein Theorem. Let $C = \{a_1, a_2, a_3, \ldots\}$ be a denumerable subset of $(0, 1]$ such that

$$1 = a_1 > a_2 > a_3 > \cdots > a_n > \cdots > 0.$$ For example, we could set $a_n = \frac{1}{2^{n-1}}$. Although $C$ is denumerable, the set $(0, 1]$ is uncountable. Define $h : (0, 1] \to (0, 1)$ by

$$h(x) = \begin{cases} a_{n+1} & \text{if } x = a_n \in C, \\ x & \text{if } x \in (0, 1] - C. \end{cases}$$
The function $h$ maps the set $C = \{a_1, a_2, a_3, \ldots\}$ bijectively to the set $D = \{a_2, a_3, a_3, \ldots\}$, and $h$ acts as the identity function on the set $(0, 1) - C = (0, 1) - D$. So, $h$ is a bijection.

A figure depicting this on the real number line is given below:

A geometric way to think about this is that the mapping $h$ sends most points in $(0, 1]$ to themselves, but when it sends the set $C = \{a_1, a_2, a_3, \ldots\}$ to the set $D = \{a_2, a_3, a_3, \ldots\}$ a ‘hole’ is created where $a_1 = 1$ used to be. This ‘converts’ the half-open, half-closed interval $(0, 1]$ into the open interval $(0, 1)$. □

**Example 4.** Show that $|\mathbb{R}| = |\mathbb{R} - \{7, 8\}|$.

**Solution.** First, since $\mathbb{R} - \{7, 8\} \subseteq \mathbb{R}$, the identity mapping is an injection from $\mathbb{R} - \{7, 8\}$ into $\mathbb{R}$. So,

$$|\mathbb{R} - \{7, 8\}| \leq |\mathbb{R}|.$$ 

Next, by Theorem 10.13 in the textbook we already know that $|(0, 1)| = |\mathbb{R}|$. Since $(0, 1) \subseteq \mathbb{R} - \{7, 8\}$, we have $|(0, 1)| \leq |\mathbb{R} - \{7, 8\}|$, and so $|\mathbb{R}| = |(0, 1)| \leq |\mathbb{R} - \{7, 8\}|$.

Since $|\mathbb{R} - \{7, 8\}| \leq |\mathbb{R}|$ and $|\mathbb{R}| \leq |\mathbb{R} - \{7, 8\}|$, the Schröder-Berstein Theorem implies that $|\mathbb{R}| = |\mathbb{R} - \{7, 8\}|$. □

**Example 5.** Show that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

**Proof.** First, we’ll show that $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$. Define

$$g: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}$$

as follows: Given a subset $S \subseteq \mathbb{N}$ let

$$g(S) = s = 0.s_1s_2s_3\ldots \quad \text{ (base 10 decimal)}$$

where

$$s_i = \begin{cases} 
0 & \text{if } i \notin S, \\
1 & \text{if } i \in S. 
\end{cases}$$

For example,

$$g(\{1, 3, 4, 6, 8, 9, \ldots\}) = 0.101101011\ldots.$$ 

To see that $g$ is injective, let $S, T \in \mathcal{P}(\mathbb{N})$ and assume $g(S) = g(T)$. Then

$$s = 0.s_1s_2s_3\ldots = t = 0.t_1t_2t_3\ldots.$$ 

Since there are no repeating 9’s, we then know $s_i = t_i$ for all $i \in \mathbb{N}$. This implies that $i \in S$ if and only if $i \in T$. Thus $S = T$. Since $g: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}$ is an injection,

$$|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|.$$
Second, we’ll show $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$. Since $|\mathbb{R}| = |(0, 1)|$, it suffices to find an injective function $f : (0, 1) \to \mathcal{P}(\mathbb{N})$. We write an element $a \in (0, 1)$ using base 2, $a = 0.a_1a_2a_3\ldots[2]$ (but we don’t allow repeating 1’s at the end). For example,

$$0.1011001\ldots[2] = \frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{1}{2^7} + \cdots$$

Define

$$f : (0, 1) \to \mathcal{P}(\mathbb{N})$$

by

$$f(a) = A$$

where

$$A = \{i : a_i \neq 0\}.$$

For example,

$$f(0.1011001\ldots[2]) = \{1, 3, 4, 7, \ldots\}.$$ 

We want to see that $f$ is an injection. Let $a, b \in \mathbb{R}$ and assume $f(a) = f(b)$. Write $a = 0.a_1a_2a_3\ldots[2]$ and $b = 0.b_1b_2b_3\ldots[2]$. Then $A = \{i : a_i \neq 0\} = B = \{i : b_i \neq 0\}$. Thus $a = b$. □

2. Proof of the Schröder-Bernstein Theorem

**Notation and Facts.** Let $A$ and $B$ be nonempty sets, and let $f : A \to B$ be a function.

- a) If $C \subseteq A$, the set $f(C)$ is defined by $f(C) = \{f(x) : x \in C\}$.
- b) If $C \subseteq A$ and $f(C) \subseteq D$, then we can get a new function $f : C \to D$ by restricting the domain and altering the codomain of the original function $f : A \to B$. [This use of notation makes sense if you keep track of the context in which the symbol $f$ is used and it makes writing a proof of the Schröder-Bernstein Theorem easier. Please don’t do this on homework or exams. Instead, use a new function name if you change the domain or codomain, which is the convention used in the textbook.]
- c) If $f : A \to B$ is injective, then the new function $f : A \to f(A)$ is a bijection since any function is surjective onto its image.
- d) If $B \subseteq A$ and if $f : A \to B$ is a function, then $x \in A$ implies $f(x) \in B \subseteq A$. So, expressions like $f(f(x))$ and $f(f(f(x)))$ make sense. For $x \in A$, we write

$$f(x) = f^1(x)$$

$$f(f(x)) = f^2(x)$$

$$f(f(f(x))) = f^3(x)$$

$$\vdots$$

$$f(f^n(x)) = f^{n+1}(x).$$

That is,

$$f^n = f \circ \cdots \circ f, \quad \text{n times}$$

Before proving the Schröder-Bernstein Theorem, we’ll prove a slightly weaker theorem.
**Theorem 1.** Suppose $A$ and $B$ are nonempty sets and that $B \subseteq A$. If $f: A \to B$ is injective, then there exists a bijection $h: A \to B$.

Note that since $B \subseteq A$, the identity map is an injection from $B$ into $A$, and so $|B| \leq |A|$. On the other hand, since $f: A \to B$ in an injection, we have $|A| \leq |B|$. Theorem 1 is a special case of the the Schröder-Bernstein Theorem in which we have the extra assumption $B \subseteq A$. In the general theorem, neither $A$ nor $B$ is assumed to be a subset of the other.

**Proof of Theorem 1.** Assume that $A$ and $B$ are nonempty sets, that $B \subseteq A$, and that the function $f: A \to B$ is injective.

If $A = B$, then $|A| = |B|$ and there is nothing to prove. If $f(A) = B$, then the injective map $f$ is also surjective, and there is nothing to prove since $f$ is already a bijection. So, in the remainder of the proof we will assume that $B \subset A$ and $f(A) \subseteq B$. Consequently, 

$$A - B \neq \emptyset \quad \text{and} \quad B - f(A) \neq \emptyset. \quad (1)$$

We’ll iteratively apply the function $f$ to the set $A - B$ to construct a special subset $C$ of $A$. Let $C$ be defined by

$$C = (A - B) \cup f(A - B) \cup f^2(A - B) \cup f^3(A - B) \cup \cdots$$

$$= (A - B) \cup \bigcup_{n \in \mathbb{N}} f^n(A - B).$$

Also let $D$ be the set

$$D = f(C) = f(A - B) \cup f^2(A - B) \cup f^3(A - B) \cup \cdots$$

$$= \bigcup_{n \in \mathbb{N}} f^n(A - B).$$

Since $C \subseteq A$ and $D = f(C)$ and since $f: A \to B$ is injective, the mapping

$$f: C \to D$$

is a bijection. Observe that

$$A - C = D.$$

The set $C$ is the union of all the shaded regions in the following Venn diagram. The set $D$ consists of the union of the shaded circles that are inside of $B$. 

![Venn Diagram](image-url)
As an exercise, you are encouraged use the injectivity of \( f: A \to B \) to verify that if \( 0 < m < n \), then \( f^m(A - B) \cap f^n(A - B) = \emptyset \). This fact and the set inequalities in equation (1) show that the Venn diagram accurately depicts the inclusion and containment relationships among the various sets.

Now we define a mapping \( h: A \to B \) by
\[
h(x) = \begin{cases} 
  f(x) & \text{if } x \in C, \\
  x & \text{if } x \in A - C = B - D.
\end{cases}
\]

This function maps \( C \) bijectively to \( D \) and maps \( A - C \) bijectively to \( B - D \). So, \( h \) is a bijection from \( A = C \cup (A - C) \) to \( B = D \cup (B - D) \). \( \square \)

Now that we have proved Theorem 1, we will prove the Schröder-Bernstein Theorem.

**Theorem 2** (Schröder-Bernstein). Let \( A \) and \( B \) be nonempty sets such that \( |A| \leq |B| \) and \( |B| \leq |A| \). Then \( |A| = |B| \).

**Proof.** By hypothesis, there exist injective mappings
\[
f: A \to B \\
g: B \to A.
\]
Then we have set inclusions \( f(A) \subseteq B \) and \( g(f(A)) \subseteq g(B) \subseteq A \) as illustrated in the figure below:

So, we have the following
\[
f: A \to f(A) \text{ is bijective.} \\
g: f(A) \to g(f(A)) \text{ is bijective.} \\
g \circ f: A \to g(f(A)) \text{ is bijective.} \\
g \circ f: A \to g(B) \text{ is injective (expanded codomain).}
\]
Since \( g \circ f: A \to g(B) \) is injective and \( g(B) \subseteq A \), Theorem 1 shows that there exists a bijection
\[
h: A \to g(B).
\]
Also we have
\[
g: B \to g(B) \text{ is bijective.} \\
g^{-1}: g(B) \to B \text{ is bijective.}
\]
Then the composition of bijective maps $h: A \to g(B)$ and $g^{-1}: g(B) \to B$ gives a bijection $g^{-1} \circ h: A \to B$, proving that $|A| = |B|$.