



TROPICAL DERIVATIVES AND DUALITY

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Introduction

Tropical Geometry is a relatively new branch of mathematics that has drawn a lot of attention. It relates to many other fields of mathematics such as combinatorics and algebraic geometry, and other sciences such as biology (e.g. phylogenetics). [SS04] Most of our work has been to solidify basic understanding of tropical mathematics and to prove many other fundamental results.

What is Tropical Geometry

In Tropical Geometry, we are dealing with the semiring

$$\mathcal{R} = (\mathbb{R} \cup \infty, \oplus, \odot),$$

where \oplus and \odot are defined by

$$a \oplus b = \min\{a, b\}$$

$$a \odot b = a + b$$

A polynomial written as $f(x) = a_0 \oplus a_1x \oplus a_2x^2 \oplus \dots \oplus a_{n-1}x^{n-1} \oplus a_nx^n$ means that

$$f(x) = \min\{a_0, a_1 + x, a_2 + 2x, \dots, a_{n-1} + (n-1)x, a_n + nx\}$$

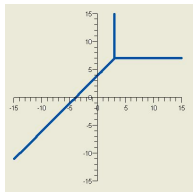


FIGURE 1: $y \oplus 4x \oplus 7$

The corner locus (denoted $Z(f)$) of a function is the set of points where the minimum is achieved twice. So for $f(x, y) = y \oplus 4x \oplus 7$, we say that $Z(f) = \{(x, y) \text{ such that } y = 4 + x < 7; y = 7 < 4 + x; 4 + x = 7 < y, \text{ which is Figure 1 above.}\}$

The Tropical Derivative

The tropical derivative is different from the classical derivative. The motivation for the following definition comes from differentiating Puiseux series and also by observing tropical lines like Figure 1.

We define

$$\frac{d}{dx} x^n := x^{n-1}$$

and extend linearly

Tropical derivatives

1. are Linear
2. satisfy the Product Rule
3. satisfy the Chain Rule

For tropical polynomials with more than one variable, we define partial derivatives in a similar manner.

Singularities in Conics

We have proven a tropical analog of a widely known theorem in classical algebraic geometry.

Theorem. A conic is factorable iff it has a singular point.

This is not as obvious as it may first seem because the polynomial semiring $\mathcal{R}[x, y]$ does not have unique factorization.

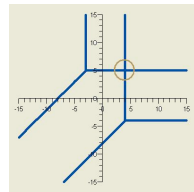


FIGURE 2: Conic $Z(y^2 \oplus -8xy \oplus -4yz \oplus x^2 \oplus -3xz \oplus 1z^2)$ with a singularity

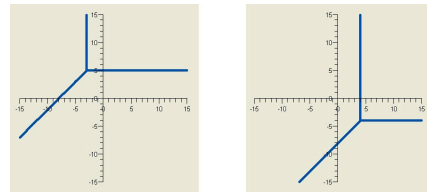


FIGURE 3: Factors of the conic in Figure 2

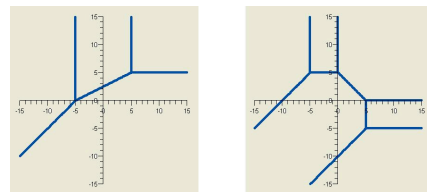


FIGURE 4: Non-Singular Conics

Tropical Duality

Classically, the set of all lines in \mathbb{P}^2 is denoted \mathbb{P}^{2*} . Each line \mathcal{L} is defined by a form

$$a_0x_0 + a_1x_1 + a_2x_2 = 0$$

and so we can identify \mathbb{P}^{2*} with \mathbb{P}^2 by the rule

$$\mathcal{L} \mapsto (a_0, a_1, a_2)$$

For an algebraic curve C defined in \mathbb{P}^2 , the dual curve C^* of C is the set of tangents of C in \mathbb{P}^{2*} . [BK]

We can do something similar with the tropical projective plane \mathbb{TP}^2 and tropical plane curves. We define the tangent of a homogenous polynomial f at (p_0, p_1, p_2) to be the line defined by the partial derivatives at (p_0, p_1, p_2) .

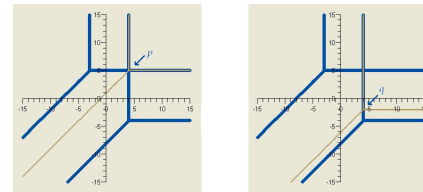


FIGURE 5: Here is a conic which is the blue graph. The brown lines are tangents to the conic at points p and q respectively.

In classical projective geometry, given a plane curve C , there is a dual curve C^* such that:

- For all points $p \in C$, the tangent to C at p is in C^*
- For all points $p^* \in C^*$, the tangent to C^* at p^* is in C

In \mathbb{TP}^2 , using the same definition will give us many candidates for dual curves. However, we have proven the following theorem.

Theorem. For every tropical plane curve C in \mathbb{TP}^2 , there exists a unique maximal tropical curve in C^* in \mathbb{TP}^{2*} which is dual to C .

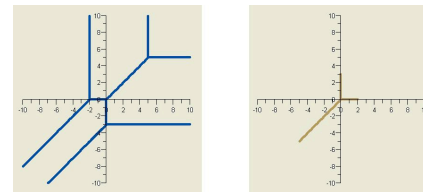


FIGURE 6: Conic $\mathcal{H} = Z(3y^2 \oplus 2x^2 \oplus xz \oplus yz \oplus xy \oplus 5z^2)$ in \mathbb{TP}^2 and its dual in \mathbb{TP}^{2*}

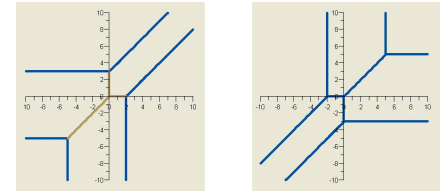


FIGURE 7: Quartic \mathcal{K} in \mathbb{TP}^2 and its dual in \mathbb{TP}^{2*}

Additional Results

Theorem. The dual curve is a tropical algebraic curve.

Theorem. C^* is the 180° rotation of C .

Corollary. $C^{**} = C$.

Tropical Plücker's Formula

In Figure 6 and 7, the dual curve of the conic \mathcal{H} was a quartic. Classically, Plücker's Formula gives us the relation between the degree of a curve and the degree of its dual curve. We have a similar formula for tropical dual curves.

For a homogenous polynomial of degree n ,

$$f = a_1x^{r_1}y^{s_1}z^{t_1} \oplus a_2x^{r_2}y^{s_2}z^{t_2} \oplus \dots \oplus a_kx^{r_k}y^{s_k}z^{t_k}$$

its dual curve has degree

$$\max\{r_1, r_2, \dots, r_k\} + \max\{s_1, s_2, \dots, s_k\} + \max\{t_1, t_2, \dots, t_k\} - n$$

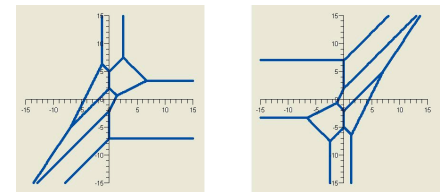


FIGURE 8: Quintic with a Degree-9 Dual Curve

References

- [BK] Egbert Brieskorn and Horst Knörrer, *Plane algebraic curves*, Birkhäuser.
 [Eil05] Amanda Ellis, *Classification of conics in the tropical projective plane*, Master's thesis, Brigham Young University, 2005.
 [SS04] David Speyer and Bernd Sturmfels, *Tropical mathematics*, Clay Mathematics Institute Senior Scholar Lecture (2004).