Factoring Tropical Polynomials in One and Two Variables

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One-variable polynomials

What is Tropical Algebra?

Tropical Algebra is the study of polynomials over the rational tropical semi-ring $\mathcal{Q} = (\mathbb{Q} \cup \infty, \oplus, \odot)$, where

$$a \oplus b = \min(a, b), \text{ and}$$
$$a \odot b = a + b.$$

It is the study of polynomials, where the "sum" of two numbers is their minimum and the "product" of two numbers is their sum.

Examples of Tropical Operations

$1 \oplus 2 = 1$	$1 \odot 2 = 3$	$x \oplus 0 = \min(x, 0)$
$2 \oplus 2 = 2$	$2 \odot 2 = 4$	$3 \odot x = x + 3$
$3 \oplus \infty = 3$	$3 \odot 0 = 3$	$1 \odot x^2 = 2 \cdot x + 1$

Polynomial Functions

- A monomial $b \odot x^a$ corresponds to the line $a \cdot x + b$.
- A one-variable polynomial is the minimum of these lines.
- Different polynomials can correspond to the same function.

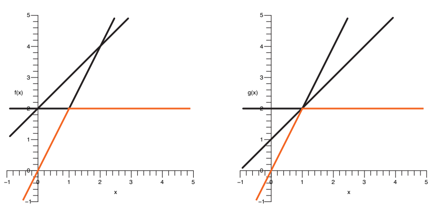


Figure 1: **Left:** $x^2 \oplus (2 \odot x) \oplus 2$; **Right:** $x^2 \oplus (1 \odot x) \oplus 2$. The black lines correspond to the monomials and the red line to their minimum. The polynomials are different but their functions are the same.

Fundamental Theorem of Tropical Algebra

- The classical *Fundamental Theorem of Algebra* states that every single-variable polynomial (over $\mathbb{C}$) can be uniquely written as the product of linear factors.
- This is also true tropically (over $\mathcal{Q}$).

Fundamental Theorem of Tropical Algebra

Every tropical polynomial in $\mathcal{Q}[x]$ can be factored uniquely into linear factors.

Sketch of Proof:

- Often two different tropical polynomials define the same function, but for every function there is a unique polynomial with smallest coefficients, called a *least-coefficient* polynomial. Every tropical polynomial is equivalent to a *least-coefficient* polynomial.
- Least-coefficient polynomials to be easily factored.

Factoring: an example

Let's factor the following polynomial (shown in Figure 2):

$$f(x) = x^3 \oplus 4x^2 \oplus 2x \oplus 4$$

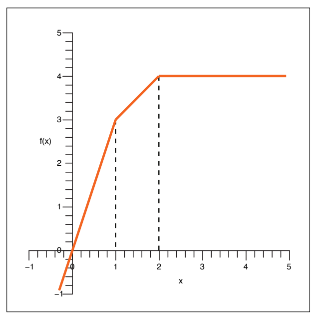


Figure 2: Graph of $f(x)$. Note that the "roots" of the polynomial correspond to the corners of the graph.

1. Find a least coefficients polynomial. The least-coefficient polynomial $x^3 \oplus 1x^2 \oplus 2x \oplus 4$ is equivalent to $f(x)$.

2. Find the difference between consecutive coefficients. We have

Monomials	Difference
$x^3, 1x^2$	$1 - 0 = 1$
$1x^2, 2x$	$2 - 1 = 1$
$2x, 4$	$4 - 2 = 2$

Notice that these differences are non-decreasing.

3. Factor the polynomial using these differences. The final result is

$$f(x) = (x \oplus 1)(x \oplus 1)(x \oplus 2).$$

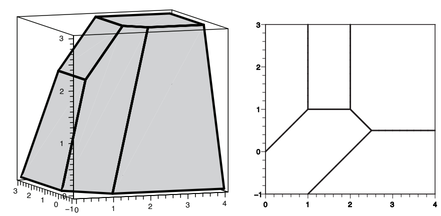
Two-variable polynomials

The Tropical Corner Locus

Definition. The *tropical corner locus* of a polynomial $f \in \mathcal{Q}[x, y]$ is the set

$$\{(x, y) : \text{two or more monomials of } f \text{ attain the minimum at } (x, y)\}.$$

- The tropical corner locus is also the set of points for which $f(x, y)$ is not smooth. See Figure 3.
- The tropical corner locus is analogous to the classical zero locus.
- The corner locus of a product is the union of the corner loci of the factors.



$$f(x, y) = x^2 \oplus xy \oplus 2y^2 \oplus 1x \oplus 3$$

Figure 3: **Left:** The surface $z = f(x, y)$. **Right:** The corner locus of $f(x, y)$.

The Weighted Balanced Graph

- A **tropical graph** is a collection of (possibly infinite) line segments (edges) that have rational slope and intersect only at their endpoints. Endpoints of these segments are called vertices.
- A **weighted graph** is a tropical graph in which each edge is assigned a natural number called its weight.
- An **integer covector** of an edge is an integer vector normal to the edge with relatively prime entries.
- Let v be a vertex of a weighted graph and $\{e_1, \ldots, e_k\}$ be the set of edges adjacent to v. For each e_i let c_i and ω_i be the corresponding integer covector and weight. Then the weighted graph is **balanced** at v if

$$\sum_{i=1}^{k} \omega_i c_i = 0.$$

- A **weighted balanced graph** is a weighted graph that is balanced at each of its vertices.

Correspondence

- There is a one-to-one correspondence between weighted balanced graphs and tropical polynomials, up to functional equivalence and multiplication by a scalar.
- The weighted balanced graph of a polynomial is just its corner locus with certain weights assigned to the edges.

Factoring Theorem

Factors of a polynomial are in correspondence with balanced subgraphs of the polynomial's weighted balanced graph.

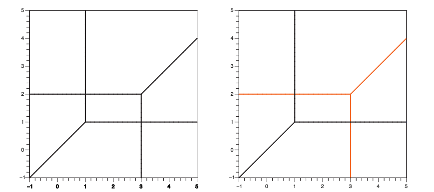
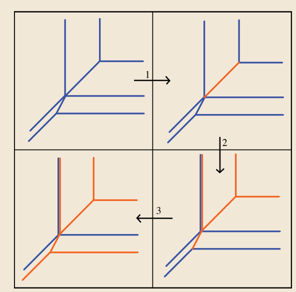


Figure 4: **Left:** A reducible cubic. **Right:** The subgraph highlighted in red is balanced and therefore corresponds to a factor.

Factoring Algorithm

I have shown that tropical factoring is NP-complete, i.e. it is highly unlikely that there is an algorithm that is efficient for all polynomials. The following algorithm is efficient for many polynomials.

1. Choose an edge e of G. Let $S = \{e\}$.
2. For each element e_i of S:
   (a) Analyze the endpoints of e_i to determine if and how each vertex can be decomposed.
   (b) If necessary, choose one of the decompositions.
   (c) Add all the edges of the decomposition to S.
3. Repeat Step 2 until no new edges are added to S.



Given the weighted balanced graph of a polynomial f, this algorithm finds a balanced subgraph, which corresponds to a factor of f.