Recovering Tropical Polynomials from Balanced Graphs

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In tropical geometry, we consider the semiring $\mathcal{R} = (\mathbb{R} \cup \infty, \oplus, \odot)$, where $\oplus$ and $\odot$ are defined by $a \oplus b = \min\{a, b\}$, $a \odot b = a + b$. A polynomial written as $f(x) = a_0 \oplus a_1 x \oplus a_2 x^2 \oplus \ldots a_{n-1} x^{n-1} \oplus a_n x^n$ means that $f(x) = \min\{a_0, a_1 + x, a_2 + 2x, \ldots a_{n-1} + (n-1)x, a_n + nx\}$. There are 2 definitions of tropical curves. One definition focuses on corner loci of tropical polynomials and the other on balanced graphs. It is easy to see that the corner locus of of a tropical polynomial is always a balanced graph. We show the converse–that every balanced comes from a tropical polynomial. We also show how to construct the polynomial explicitly.

From a Polynomial to a Balanced Graph

Corner Loci

Definition. Given a tropical polynomial $f(x_1, \ldots, x_n)$, the *corner locus* of f is defined to be the set $Z(f)$ is the set of point where at least two monomials of f attain the minimum value simultaneously.

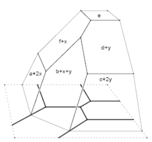
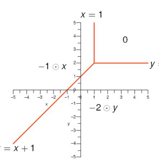


FIGURE 1: $ax^2 \oplus bxy \oplus cy^2 \oplus dy \oplus fx \oplus e$ [1]

Figure 1 shows the intersections of the planes, the classical form of each monomial, projected into $\mathbb{R}^2$ make up the corner locus of the polynomial.



The graph above of $f(x, y) = -2y \oplus -1x \oplus 0$ is obtained from the following inequalities:

$$-2 + y = -1 + x \leq 0 \quad -2 + y = 0 \leq -1 + x \quad -1 + x = 0 \leq -2 + y$$

and we get $y = x + 1$, $y = 2$ and $x = 1$.

Each region in the complement of the corner locus of a polynomial corresponds to the monomial that achieves the minimum on that entire region.

First Definition of Tropical Curve. A tropical curve is a subset of $\mathbb{R}^2$ that is the corner locus of a tropical polynomial.

Balanced Graphs

Definition. A *weighted graph* is a graph such that each pair of a vertex and an incident edge has a corresponding vector $v_i = \omega_i \alpha_i$. α_i is called a primitive integer vector. ω_i is a natural number that is called a weight.

Definition. A weighted graph is *balanced* if $\sum \omega_i \alpha_i = 0$ around each vertex. This implies that the $\sum \omega_i \alpha_i = 0$ around the entire graph.

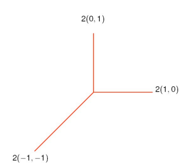


FIGURE 2: A balanced graph

Figure 2 is an example of a balanced graph and Figure 3 is an example of a weighted graph that is not balanced.

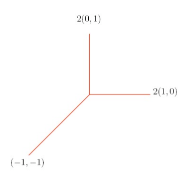


FIGURE 3: A non-balanced graph

Second Definition of Tropical Curve. A tropical curve is a balanced graph Γ in $\mathbb{R}^2$ such that every line segment in Γ has rational slope.

From a Balanced Graph to a Polynomial

Here we outline an algorithm to recover tropical polynomials whose corner locus is a given balanced graph.

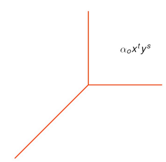


FIGURE 4: Arbitrary monomial assigned to a region.

For each region in the compliment of a corner locus there is monomial that attains the minimum there. The algorithm is based on using the regions of the balanced graph to recover a monomial that achieves the minimum for each region. First we assign an arbitrary monomial to a region as in Figure 4.

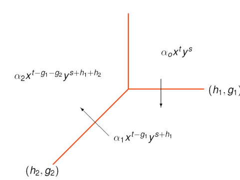


FIGURE 5: Recovering other monomials for corresponding regions.

Constructing the exponents for the monomial in the next counterclockwise region is demonstrated in Figure 5. Recall that a line segment in a corner locus is where the monomials corresponding to the adjacent regions attain the minimum together.

$$\alpha_0 x^t y^s = \alpha_1 x^{t-g_1} y^{s+h_1}.$$

Classically this is equivalent to

$$\alpha_0 + tx + sy = \alpha_1 + (t - g_1)x + (s + h_1)y \Longrightarrow y = \frac{g_1}{h_1}x + \frac{\alpha_1 - \alpha_0}{h_1}$$

which is the line segment desired.

Once this is done for each region, the coefficients α_i must be solved for. Monomials for adjacent regions are equal on the line segment dividing them. By evaluating each monomial for adjacent regions at a vertex we get a system of linear equations in terms of the α_i. The (x_i, y_i) are vertices.

$$j^{th} \left[\begin{array}{cccccc|c} 1 & -1 & 0 & \ldots & 0 & 0 & \langle (h_1, g_1), (y_1, -x_1) \rangle \\ 0 & 1 & -1 & 0 & \ldots & 0 & \langle (h_2, g_2), (y_2, -x_2) \rangle \\ \vdots & & & & & \vdots & \vdots \\ 0 & \ldots & 1 & -1 & \cdots & 0 & \langle (h_{j+1}, g_{j+1}), (y_{j+1}, -x_{j+1}) \rangle \\ \vdots & & & & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & -1 & \langle (h_{n-1}, g_{n-1}), (y_{n-1}, -x_{n-1}) \rangle \end{array}\right]$$

This system of linear equations is under-determined and thus the corresponding matrix has a non-trivial solution.

After solving for the coefficients the polynomial is completely reconstructed. In order to prove that the corner locus of the recovered polynomial is the original balanced we evaluate each monomial at a point from its designated region and show it must achieve the minimum there.

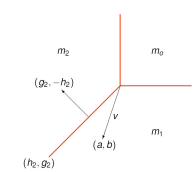


FIGURE 6: $\langle v, (g_2, -h_2) \rangle < 0$ shows (a, b) in region corresponding to m_1

As Figure 6 shows for the monomial m_1 we choose a point (a, b). We know $\langle v, (g_2, -h_2) \rangle < 0$. With a few arithmetic steps this inequality implies that $m_1 < m_2$ for every point in the region corresponding to m_1. Following this same pattern for every monomial we see that each monomial achieves the minimum on its corresponding region. Therefore the corner locus of the recovered polynomial is the same as the original balanced graph.

References

[1] Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald. First steps in tropical geometry. In *Idempotent mathematics and mathematical physics*, volume 377 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 2005.