

## CHAPTER 2: A LITTLE SET THEORY

### SET THEORY

By a set we shall mean a collection of objects, without repeats, which we shall call the elements of the set. A set can consist of any objects; numbers, people, countries, etc could be the elements or members of a set. A set can consist of a finite number or an infinite number of elements or **no elements at all**. We shall usually use capital roman letters to denote sets. Here are some examples:

- (1)  $A = \{\text{Students in Math 190}\}$ ,
- (2)  $B = \{1, 2, 5, 7\}$ ,
- (3)  $C = \{1, 2, 3, \dots, 100\}$ ,
- (4)  $\mathbb{N} = \{1, 2, 3, \dots\} = \{\text{Natural numbers}\}$ .

If  $a$  is an element of some set  $S$  then we say  $a$  belongs to  $S$  or  $a$  is a member of  $S$  and we write  $s \in S$ . Which we read as “ $a$  is an element of  $S$ ”.

The set which has no elements we call the empty set and we use the symbol  $\emptyset$  to denote it.

If  $A$  and  $B$  are sets and every element of  $A$  is also an element of  $B$  we say that  $A$  is a subset of  $B$  and use the notation  $A \subset B$ . This can also be written  $B \supset A$  which we read as  $B$  is a superset of  $A$ . If  $A$  and  $B$  have precisely the same elements we say that they are equal and denote them by  $A = B$ .

**Theorem.** *If  $A$  and  $B$  are sets then  $A = B$  if and only if  $A \subset B$  and  $B \subset A$ .*

We use the notation  $A = \{a : \text{Property } P \text{ holds for } a\}$  to denote the set of elements  $a$  for which property  $P$  holds. If we are only interested in considering elements which have property  $P$  and also lie in the set  $B$  we can write this as  $A = \{a \in B : \text{Property } P \text{ holds for } a\}$ .

If  $A$  and  $B$  are sets then the elements which lie in both  $A$  and  $B$  form a set called the intersection of  $A$  and  $B$ . We use the notation  $\cap$  to denote intersection. So  $C = A \cap B$  means that  $C$  is the set which consists of precisely those elements of  $A$  which are also elements of  $B$ .

#### Examples.

- (1)  $A = \{2, 5, 7, 11\}$ ,  $B = \{5, 6, 7, 23, 35, 101\}$  then

$$A \cap B = \{5, 7\}.$$

- (2)  $F = \{\text{Females}\}$ ,  $S = \{\text{Math190 students}\}$  then

$$F \cap S = \{\text{Female Math190 students}\}.$$

If  $A$  and  $B$  are sets then the elements which lie in either  $A$  or  $B$  is called the union of the sets  $A$  and  $B$  and is written as  $A \cup B$ . In example 1 above

$$A \cup B = \{2, 5, 6, 7, 11, 23, 35, 101\}.$$

In set theory it is necessary to postulate the existence of a Universal set  $\mathcal{U}$  which contains all of the elements of the sets which we are considering. This is necessary to avoid the logical pitfall brought about by the following paradoxical example due to Bertrand Russell and known as

**Russell's paradox.**

Let  $S$  be the set of all sets that do not contain themselves as elements.

Can  $S$  exist ?

In order to avoid this problem we limit the size of the collection of sets being considered by referring to some Universal set in which all sets and elements are considered to lie. We are thereby restricting the ability to construct something like: "The set of all sets."

By the complement of a set  $A$  we mean all elements in our Universal set which do not lie in  $A$ .

Thus in the integers the complement of  $A = \{2, 5, 6\}$  would be  $B = \{1, 3, 4, 7, 8, 9, \dots\}$  where by the  $\dots$  we understand that the sequence extends to all natural numbers greater than 9. We denote the complement of a set  $A$  by  $\complement A$ . Sometimes we will wish to refer to the elements of some set  $C$  which do not lie in a set  $D$ . We call this set the complement of  $D$  in  $C$  and use the notation  $C \setminus D$ . That is

$$C \setminus D = C \cap \complement D.$$

**Definition of Cartesian product.** If  $A$  and  $B$  are sets, define the Cartesian product of  $A$  and  $B$ , written  $A \times B$ , to be the set of all ordered pairs  $(a, b)$  such that  $a \in A$  and  $b \in B$ .

**Exercises** Let  $A, B$  and  $C$  be sets.

- (1) Show that  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .
- (2) Show that  $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ .
- (3) State similar results for  $A \cap (B \cup C)$  and for  $A \setminus (B \cap C)$ .

## BINARY RELATIONS

Let  $S$  be a set then a binary relation on  $S$  is a subset of the set of pairs  $(x, y)$  where  $x$  and  $y$  are elements of  $S$ .

**Definition: Equivalence relations.** If  $S$  is a set with a binary relation  $\sim$  defined on it the  $\sim$  is an equivalence relation on  $S$  if:

- (1) For all  $x \in S$   $x \sim x$  (reflexive property).
- (2) If  $x, y \in S$  and  $x \sim y$  then  $y \sim x$  (symmetric property).
- (3) If  $x, y, z \in S$ ,  $x \sim y$  and  $y \sim z$  then  $x \sim z$  (transitive property).