THE DISTRIBUTION OF k-FREE NUMBERS

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Abstract. Let $k \in \{3, 4, 5\}$. Let

$$R_k(x) = \sum_{\substack{n \le x \\ n \text{ is } k\text{-free}}} 1 - \frac{x}{\zeta(k)}.$$

We give new upper bounds for $R_k(x)$ conditional on the Riemann hypothesis, improving work of S.W. Graham and J. Pintz. The method stays close to that devised by H.L. Montgomery and R.C. Vaughan, with the improvement depending on exponential sum results.

1. Introduction

Let $k \ge 2$. A positive integer n is said to be k-free if n is not divisible by the k-th power of a prime. Let

$$R_k(x) = \sum_{\substack{n \le x \\ n \text{ is } k\text{-free}}} 1 - \frac{x}{\zeta(k)}.$$

An elementary argument yields

$$R_k(x) \ll x^{1/k}.$$

In the opposite direction,

$$R_k(x) = \Omega(x^{1/2k}).$$

(See [5] or [15]).

Assuming the Riemann hypothesis (RH), Montgomery and Vaughan [14] obtained

$$R_k(x) \ll x^{1/(k+1)+\epsilon}. \tag{1.1}$$

Here ϵ is an arbitrarily small positive number. The exponent in (1.1) has been improved for every value of k. For $k \ge 3$, see Graham and Pintz [9]. For $k = 2$, there have been papers by Graham [7], Baker and Pintz [4] and Jia [13]; the exponent in [13] is $17/54 + \epsilon$.

No-one has yet been able to improve the approximation in Theorem 1 of [14] under RH: for $N \ge 1$,

$$R_k(x) = -\sum_{n \le N} \mu(n)\psi(xn^{-k}) + \mathrm{O}(x^{1/2+\epsilon}N^{(1-k)/2} + N^{1/2+\epsilon}). \tag{1.2}$$

Here $\psi(w) = w - [w] - 1/2$. Once (1.2) is applied, exponential sums dominate the discussion.

Jia's paper, which appeared after [9], contains an estimate for exponential sums which has found several applications. Abstractions of the estimate are given by Wu [18]. The exponential sum estimate devised by Heath-Brown [12] is crucial in [4, 9, 13]; it is abstracted in Baker [1]. In the present paper we combine these results with theorems from Robert and Sargos [16] and Baker [2]. The former paper is a natural culmination of the work of Fouvry and Iwaniec [6]; the starting point is the double large sieve. The latter paper is related to earlier work of Baker and Kolesnik [3]. Actually, we shall adapt the results from [2, 18] a little.

Theorem. *Assume RH. We have*

$$R_k(x) \ll x^{\theta_k+\epsilon} \quad (k=3,\ 4,\ 5)$$

where

$$\theta_3 = \frac{17}{74} = 0.2297...,\ \theta_4 = \frac{17}{94} = 0.1808...,\ \theta_5 = 0.15.$$

For comparison, θ_k replaces $7/(6k+8)$ in [9], and we have

$$\frac{7}{30} = 0.2333...,\ \frac{7}{38} = 0.1842...,\ \frac{7}{46} = 0.1521....$$

In contrast with [14, 7, 4, 9, 13], we apply Heath-Brown's "generalized Vaughan identity" [12], or more precisely, the variant provided in [2]. We add a new result on the application of this decomposition (the case $\frac{1}{6} < a \leq \frac{1}{5}$ of Lemma 3(iii) below) which may be of independent interest.

We close this section with a few remarks on notation. Implied constants depend at most on k and ϵ, except in section 3 where they may depend on u, v, α, β, γ, κ and λ. We write "$A \asymp B$" if $A \ll B \ll A$. We write "$n \sim N$" for $N < n \leq 2N$. As usual, $e(z) = e^{2\pi i z}$. The cardinality of a finite set E is written $|E|$.

2. Decompostition of sums involving the Möbius function

Let $2 \leq D < D' \leq 2D$ and let f be a complex function on $[D, D']$. The sum

$$\sum_{D<n\leq D'} \mu(n) f(n)$$

can be decomposed into $O((logD)^{2l-1})$ sums of the form

$$\sum_{\substack{n_i\sim N_i \\ D<n_1...n_{2l-1}\leq D'}} \mu(n_l)...\mu(n_{2l-1}) f(n_1...n_{2l-1}). \tag{2.1}$$

Here $N_i > \frac{1}{2}$, $\prod_{i=1}^{2l-1} N_i \asymp D$ and

$$2N_i \leq (2D)^{1/l} \text{ if } i \geq l. \tag{2.2}$$

See [2, Section 2].

To apply the decomposition, we need results of the following kind. The numbers α_i $(1 \leq i \leq r)$ in Lemma 1 arise as exponents for which $N_i \asymp D^{\alpha_i}$. (To be precise, let $r = 2l-1$, $N_0 = 2^{2l-1}N_1...N_{2l-1}$ and define α_i by

$$2N_i = N_0^{\alpha_i} \text{ for } 1 \leq i \leq 2l-1.)$$

Lemma 1. *Let $0 \leq \alpha_1 \leq ... \leq \alpha_r$, $\alpha_1 + ... + \alpha_r = 1$. For $S \subseteq \{1, ..., r\}$, we write $\sigma_s = \sum_{i\in S} \alpha_i$.*

(i) Let h be an integer, $h \geq 3$. Suppose that $\alpha_r \leq \frac{2}{h+1}$. Then some $\sigma_s \in [\frac{1}{h}, \frac{2}{h+1}]$.

(ii) Let $\lambda \geq \frac{2}{3}$. Suppose that $\alpha_r \leq \lambda$. Then some $\sigma_s \in [1-\lambda, \lambda]$.

Proof. See [2, Lemma 1].

The following lemma and Lemma 3(iii) are based on a result in Heath Brown [2], which it strengthens in the case $\frac{1}{6} < a \le \frac{1}{5}$.

Lemma 2. *Make the hypotheses of Lemma 1.*

(i) Let $\frac{1}{5} < a \le \frac{1}{3}$*. Suppose that* $\alpha_r \le \frac{1-a}{2}$*. Then some* $\sigma_s \in [a, 2a]$*.*

(ii) Let $0 < a \le \frac{1}{5}$*. Suppose that* $\alpha_r \le \frac{1-a}{2}$*. Then some* $\sigma_s \in [a, \frac{1}{3}]$*.*

Proof. (i) We have $\frac{1}{h+1} < a \le \frac{1}{h}$ for some natural number h, which must be 3 or 4. Now

$$\alpha_r \le \frac{1-a}{2} < \frac{2}{5} \le \frac{2}{h+1}.$$

By Lemma 1(i), some $\sigma_s \in [\frac{1}{h}, \frac{2}{h+1}] \subseteq [a, 2a]$.

(ii) Suppose that no $\sigma_s \in [a, \frac{1}{3}]$. Let

$$T = \{j : \alpha_j \le \frac{1}{3} - a\} \quad , \quad U = \{j : \frac{1}{3} - a < \alpha_j < a\}$$

(U is empty if $a \le \frac{1}{6}$), and

$$V = \{j : \alpha_j > \frac{1}{3}\}.$$

Thus

$$\sigma_T + \sigma_U + \sigma_V = 1. \tag{2.3}$$

Clearly

$$|V| \le 2 \quad , \quad \sigma_V \le |V|\left(\frac{1-a}{2}\right) \le 1 - a. \tag{2.4}$$

Suppose for a moment that U is nonempty. Then for any $j \in U$,

$$\sigma_T + \alpha_j < a. \tag{2.5}$$

To see this suppose the contrary, and take the smallest $\sigma_W \ge a - \alpha_j$ with $W \subseteq T$. Then

$$\sigma_W - \alpha_k < a - \alpha_j$$

for any $k \in W$. Hence $a \le \sigma_W + \alpha_j < a + \alpha_k \le \frac{1}{3}$, contrary to our hypothesis. A similar argument gives $\sigma_T < a$ if U is empty, and it follows that

$$\sigma_T + \sigma_U < a + \max(0, |U| - 1)a = \max(1, |U|)a. \tag{2.6}$$

Suppose for a moment that $|U| \ge 2$. Take α_i, α_j in U, $i \ne j$. Then

$$\alpha_i + \alpha_j > \frac{2}{3} - 2a > a.$$

Consequently, $\alpha_i + \alpha_j > \frac{1}{3}$. This yields

$$\text{If } |U| \ge 2, \text{ then } \sigma_U > \frac{|U|}{2}\frac{1}{3} = \frac{|U|}{6}. \tag{2.7}$$

In particular, $|U| \le 5$.

We now consider all possibilities for $|U|$.

Suppose $|U| = 0$ or 1. From (2.6),

$$\sigma_T + \sigma_U < a,$$

and from (2.4),

$$\sigma_T + \sigma_U + \sigma_V < 1,$$

which is absurd.

Suppose $|U| = 2$ or 3. Then $\sigma_U > \frac{1}{3}$ from (2.7), and so $\sigma_V < \frac{2}{3}$ and $|V| \le 1$,

$$\sigma_V \le \frac{1-a}{2}.$$

In conjunction with (2.6), this yields

$$\sigma_T + \sigma_U + \sigma_V < 3a + \frac{1-a}{2} \le 1,$$

which is absurd.

Suppose finally that $|U| = 4$ or 5. From (2.6) and (2.7),

$$\frac{2}{3} < \sigma_T + \sigma_U < 5a \le 1,$$

so that $\sigma_V \in (0, \frac{1}{3})$. This is absurd and the lemma is proved.

Lemma 3. *Let f be a complex function on $(D, D']$ where $2 \le D < D' \le 2D$.*

(i) Suppose that

$$\sum_{\substack{m\sim M \\ D<mn\le D'}} \sum_{n\sim N} a_m b_n f(mn) \ll B \tag{2.8}$$

whenever $|a_m| \le 1$, $|b_n| \le 1$ and

$$D^{1/h} \ll N \ll D^{2/(h+1)}$$

where h is a natural number, $h \ge 3$. Suppose further that

$$\sum_{\substack{m\sim M \\ D<mn\le D'}} a_m \sum_{n\sim N} f(mn) \ll B \tag{2.9}$$

whenever $|a_m| \le 1$ and

$$N \gg D^{2/(h+1)}.$$

Then

$$\sum_{D<d\le D'} \mu(d) f(d) \ll BD^{\epsilon}. \tag{2.10}$$

(ii) Let $\lambda \ge \frac{2}{3}$. Suppose that (2.8) holds whenever $|a_m| \le 1$, $|b_n| \le 1$ and

$$D^{1-\lambda} \ll N \ll D^{1/2},$$

while (2.9) holds whenever $|a_m| \le 1$ and

$$N \gg D^{\lambda}.$$

Then (2.10) holds.

(iii) Let $0 < a \le \frac{1}{3}$. Let $b = 2a$ if $a > \frac{1}{5}$ and $b = \frac{1}{3}$ if $a \le \frac{1}{5}$. Suppose that (2.8) holds whenever $|a_m| \le 1$, $|b_n| \le 1$ and

$$D^a \ll N \ll D^b,$$

while (2.9) holds whenever $|a_m| \le 1$ and

$$N \gg D^{(1-a)/2}.$$

Then (2.10) holds.

Proof. We prove (iii); the other proofs are similar. Take $l = 4$ in the decomposition into sums (2.1). We may suppose that D is large. Given one of the sums (2.1), let N_0 and $\alpha_1, ..., \alpha_{2l-1}$ have the values assigned before Lemma 1. We claim that the sum (2.1) is $O(BN^{\epsilon/2})$. If we can group the variables as

$$n = \prod_{i \in S} \quad , \quad m = \prod_{i \notin S} n_i$$

with $\sigma_S \in [a, b]$, then from (2.8) the sum in (2.1) is $O(BN^{\epsilon/2})$, since the corresponding coefficients a_m, b_n are $O(N^{\epsilon/2})$. If this grouping is not possible, then by Lemma 2 there is an $\alpha_j > \frac{1-a}{2}$. Now we group the variables as

$$m = \prod_{i \neq j} n_i \quad , \quad n = n_j.$$

Note that $j < l$ from (2.2). It follows from (2.9) that the sum in (2.1) is $O(BN^{\epsilon/2})$. Now (iii) follows at once.

3. Lemmas on Exponential Sums

We quote two preliminary lemmas from Graham and Kolesnik [8].

Lemma 4. *(Kusmin-Landau) If the real function f is continuously differentiable, and f' is monotonic with*

$$0 < \lambda \leq |f'| \leq \frac{1}{2}$$

on the interval I, then

$$\sum_{n \in I} e\Big(f(n)\Big) \ll \lambda^{-1}.$$

Proof. See [8], Theorem 2.1.

Lemma 5. *Let*

$$E(H) = \sum_{i=1}^{u} A_i H^{a_i} + \sum_{j=1}^{v} B_j H^{-b_j}$$

where A_i, B_j, a_i and b_j are positive. Let $0 \leq H_1 \leq H_2$. Then there is some $H \in (H_1, H_2]$ with

$$E(H) \ll \sum_{i=1}^{u} \sum_{j=1}^{v} ({A_i}^{b_j} {B_j}^{a_i})^{1/(a_i+b_j)} + \sum_{i=1}^{u} A_i H_1^{a_i} + \sum_{j=1}^{v} B_j {H_2}^{-b_j}.$$

Proof. This follows at once from [8], Lemma 2.4.

In the remainder of this section, we write $L = log(D + 2)$. Our first lemma is a variant of [2, Theorem 5].

Lemma 6. *Let (κ, λ) be an exponent pair. Let α, β be real constants, $\beta \neq 0$, $\beta < 1$, $\alpha < 0$. Suppose that*

$$X > 0, \frac{1}{2} \leq N \ll M, MN \asymp D, D < D' \leq 2D. \tag{3.1}$$

Let

$$S_1 = \sum_{\substack{m \sim M \\ D \leq mn < D'}} a_m \sum_{n \sim N} b_n e\left(\frac{X m^\alpha n^\beta}{M^\alpha N^\beta}\right) \tag{3.2}$$

where $|a_m| \le 1$, $|b_n| \le 1$. *Then*

$$(3.3)\quad S_1 \ll L^{7/4}\Bigg(DN^{-1/2} + DM^{-1/4} + D^{5/4}X^{-1/4}N^{-3/4} + D^{5/4}X^{-1/4}M^{-3/8} + (D^{11+10\kappa}X^{1+2\kappa}N^{2(\lambda-\kappa)})^{1/(14+12\kappa)} + (D^{5+4\kappa}N^{\lambda-\kappa})^{1/(6+4\kappa)}\Bigg).$$

Proof. In [2], proof of Theorem 5, it is shown that

$$(3.4)\qquad L^{-7}S_1^4 \ll \frac{D^4}{Q^2} + D^3\left(\left(\frac{XQ^3}{D}\right)^{1/2+\kappa} N^{\lambda-\kappa} + N + D^2X^{-1}Q^{-3}\right)$$

for any natural number $Q \le \min(c^2N, cM^{-1/2})$. Here and in the remainder of the paper, c is a small positive constant. With a little thought we see that (3.4) is true for any real Q with $0 < Q \le \min(c^2N, cM^{-1/2})$. Now Lemma 5 yields

$$L^{-7}S_1^4 \ll D^4N^{-2} + D^4M^{-1} + (D^4)^{(3+6\kappa)/(7+6\kappa)}(D^{5/2-\kappa}X^{1/2+\kappa}N^{\lambda-\kappa})^{4/(7+6\kappa)} + (D^5X^{-1})^{(3+6\kappa)/(9+6\kappa)}(D^{5/2-\kappa}X^{1/2+\kappa}N^{\lambda-\kappa})^{6/(9+6\kappa)} + D^5X^{-1}M^{-3/2} + D^5X^{-1}N^{-3}.$$

After some simplification, we obtain (3.3).

Our next lemma improves (2.2) of [18, Theorem 2]; the term $(X^{-1}M^{14}N^{23})^{1/22}$ has been dropped.

Lemma 7. *Let* α, β *be real constants,* $\alpha\beta(\alpha-1)(\beta-1) \ne 0$. *Suppose that (3.1) holds and* $|a_m| \le 1$, $|b_n| \le 1$. *The sum* S_1 *in (3.2) satisfies*

$$(3.5)\quad S_1 \ll L^3\Bigg((XM^3N^4)^{1/5} + (X^4M^{10}N^{11})^{1/16} + (XM^7N^{10})^{1/11} + MN^{1/2} + X^{-1/2}MN\Bigg).$$

Proof. Define S_0 in the same way as S_1 but without the condition $D < mn \le D'$. It suffices to prove (3.5) with S_1, L^3 replaced by S_0, L^2, as explained in Harman [10, pp. 49-50].

If $X \le M$, the double large sieve [6, Theorem 2] yields

$$S_0 \ll L\Bigg((XMN)^{1/2} + MN^{1/2} + X^{-1/2}MN\Bigg)$$

which is satisfactory since $(XMN)^{1/2} \ll MN^{1/2}$.

Suppose now that $X > M$. We follow the proof of [18, Theorem 2] to save space. Obviously we may assume that $N > L^3$, since $S_0 \ll MN \ll MN^{1/2}L^{3/2}$ otherwise. The first step (a Weyl shift) gives

$$(3.6)\qquad S_0^2 \ll \frac{(MN)^2}{Q} + LM^{3/2}NQ^{-1}|S(Q_1)|$$

where $Q \in [L, \frac{N}{L}]$ is a parameter at our disposal, and $1 \le Q_1 \le Q$. Here

$$S(Q_1) = \sum_{q_1 \sim Q_1}\sum_{n+q_1, n \sim N} b_{n+q_1}\bar{b}_n \sum_{m \sim M} m^{-1/2} e\Bigg(Am^\alpha t(n, q_1)\Bigg),$$

$$A = \frac{X}{M^\alpha N^\beta} \quad , \quad t(n, q_1) = (n+q_1)^\beta - n^\beta.$$

Suppose initially that

$$X(MN)^{-1}Q \geq c. \tag{3.7}$$

The next step (a B process) yields

$$L^{-1}S(Q_1) \ll (XM^{-1}N^{-1}Q_1)^{1/2}S^*(Q_1) \tag{3.8}$$
$$+(XM^{-1}N^{-1}Q_1^3)^{1/2} + M^{-1/2}NQ_1 + (X^{-1}MNQ_1)^{1/2} + (X^{-2}MN^4)^{1/2}.$$

Here

$$S^*(Q_1) = \sum_{n\sim N}\left|\sum_{q_1\sim Q_1} b_{n+q_1}e\Big(\theta q_1 + \tilde{\alpha}(At)^{\gamma}l^{1-\gamma}\Big)\right|,$$

θ is a real constant, $l \asymp X(MN)^{-1}Q_1$, $\gamma = \frac{1}{1-\alpha}$ and $\tilde{\alpha} = |1-\alpha||\alpha|^{\alpha/(1-\alpha)}$.

For the next step, suppose for the moment that $Q_1 \geq L$. A Weyl shift yields

$$S^*(Q_1)^2 \ll (NQ_1)^2Q_2^{-1} + NQ_1Q_2^{-1}\sum_{1\leq q_2\leq Q_2}|S_2(q_2)| \tag{3.9}$$

where Q_2 is chosen below, $Q_2 \leq c\sqrt{Q_1}$, and

$$S_2(q_2) = \sum_{n\sim N}\sum_{q_1+q_2,q_1\sim Q_1}\bar{b}_{n+q_1}b_{n+q_1+q_2}e\Big(t_1(n,q_1,q_2)\Big)$$

with

$$t_1(n,q_1,q_2) = \tilde{\alpha}A^{\gamma}l^{1-\gamma}(t(n,q_1+q_2)^{\gamma} - t(n,q_1)^{\gamma}).$$

Note that Wu uses the incorrect expression $S_2(q_1,q_2)$ in place of $S_2(q_2)$.

The final step in [18], another Weyl shift, yields

$$\left(\frac{S_2(q_2)}{L}\right)^2 \ll (NQ_1)^2Q_2^{-2} + NQ_1Q_2^{-2}\sum_{1\leq q_3\leq Q_2^2}\sum_{q_1\sim Q_1}|S_3(q_1,q_2,q_3)| \tag{3.10}$$

with

$$S_3(q_1,q_2,q_3) = \sum_{n'\sim N}e\Big(f(n')\Big)$$

for an f (depending on the q_i) with

$$f'(n') \asymp XN^{-2}Q_1^{-1}q_2q_3 \quad (n'\sim N).$$

We choose

$$Q_2 = c\min(Q_1^{1/2}, (X^{-1}N^2Q_1)^{1/3}),$$

so that

$$|f'(n')| \leq \frac{1}{2} \quad \text{for } n'\sim N$$

and

$$S_3(q_1,q_2,q_3) \ll (XN^{-2}Q_1^{-1}q_2q_3)^{-1}$$

(Lemma 4). From (3.10),

$$\left(\frac{S_2(q_2)}{L}\right)^2 \ll (NQ_1)^2Q_2^{-2} + NQ_1^2Q_2^{-2}(XN^{-2}Q_1^{-1}q_2)^{-1}\sum_{1\leq q_3\leq Q_2^2}q_3^{-1},$$

$$L^{-3}S_2(q_2)^2 \ll N^2Q_1^2Q_2^{-2} + X^{-1}N^3Q_1^3Q_2^{-2}q_2^{-1}.$$

Now (3.9) yields

$$L^{-3/2}S^*(Q_1)^2 \ll N^2Q_1^2Q_2^{-1}+NQ_1Q_2^{-1}\sum_{q_2\le Q_2}(NQ_1Q_2^{-1}+X^{-1/2}N^{3/2}Q_1^{3/2}Q_2^{-1}q_2^{-1/2})$$
$$\ll N^2Q_1^2Q_2^{-1}+N^{5/2}Q_1^{5/2}Q_2^{-3/2}X^{-1/2}$$
$$\ll N^2Q_1^{3/2}+N^{4/3}Q_1^{5/3}X^{1/3}+N^{5/2}Q_1^{7/4}X^{-1/2},$$

where we used the value of Q_2 in the last step.

Recalling (3.8),

$$L^{-7/4}S(Q_1)\ll \tag{3.11}$$
$$(XM^{-1}N^{-1}Q_1)^{1/2}(NQ_1^{3/4}+N^{2/3}Q_1^{5/6}X^{1/6}+N^{5/4}Q_1^{7/8}X^{-1/4})$$
$$+X^{1/2}M^{-1/2}N^{-1/2}Q_1^{3/2}+M^{-1/2}NQ_1+X^{-1/2}M^{1/2}N^{1/2}Q_1^{1/2}+X^{-1}M^{1/2}N^2$$
$$\ll X^{1/2}M^{-1/2}N^{1/2}Q_1^{5/4}+X^{2/3}M^{-1/2}N^{1/6}Q_1^{4/3}+X^{1/4}M^{-1/2}N^{3/4}Q_1^{11/8}+X^{-1}M^{1/2}N^2.$$

We were able to discard three of the first last four terms in the first bound in (3.11):

$$X^{1/2}M^{-1/2}N^{-1/2}Q_1^{3/2}\ll X^{1/2}M^{-1/2}N^{1/2}Q_1^{5/4}, \text{ since } Q_1<N;$$

$$M^{-1/2}NQ_1\ll X^{1/2}M^{-1/2}N^{1/2}Q_1^{5/4}, \text{ since } X^{1/2}N^{1/2}\gg N;$$

$$X^{-1/2}M^{1/2}N^{1/2}Q_1^{1/2}<X^{1/2}M^{-1/2}N^{1/2}Q_1^{5/4}, \text{ since } M<X.$$

Recalling (3.6),

$$L^{-7/4}S_0^2\ll\frac{(MN)^2}{Q}+LM^{3/2}NQ^{-1}\Big(X^{1/2}M^{-1/2}N^{1/2}Q_1^{5/4}+$$
$$X^{2/3}M^{-1/2}N^{1/6}Q_1^{4/3}+X^{1/4}M^{-1/2}N^{3/4}Q_1^{11/8}+X^{-1}M^{1/2}N^2\Big)$$

Thus,

$$L^{-7/4}S_0^2\ll \tag{3.12}$$
$$L\frac{(MN)^2}{Q}+LMN^{3/2}X^{1/2}Q^{1/4}+LMN^{7/6}X^{2/3}Q^{1/3}+LMN^{7/4}X^{1/4}Q^{3/8}.$$

Here we used

$$LM^{3/2}NQ^{-1}(X^{-1}M^{1/2}N^2)=LM^2N^3X^{-1}Q^{-1}<LMN^3Q^{-1}\ll LM^2N^2Q^{-1}.$$

The inequality (3.12) remains valid if $Q_1<L$. For then (3.6) yields trivially

$$S_0^2\ll\frac{(MN)^2}{Q}+L^2M^2N^2Q^{-1}\ll L^2(MN)^2Q^{-1}.$$

Suppose now that $X(MN)^{-1}Q<c$. We remove $m^{-1/2}$ from $S(Q_1)$ by partial summation and apply Lemma 4, since

$$\frac{d}{dm}\Big(Am^\alpha t(n,q_1)\Big)<\frac{1}{2}\quad (m\sim M)$$

for all relevant n, q_1. Now (3.6) yields

$$S_0^2\ll\frac{(MN)^2}{Q}+\frac{MN}{Q}\sum_{1\le q_1\le Q}\Big(X(MN)^{-1}q_1\Big)^{-1}$$

$$\ll \frac{(MN)^2}{Q} + LM^2N^2Q^{-1}X^{-1} \ll \frac{L(MN)^2}{Q}$$

since $M < X$. Thus (3.12) always holds, and indeed remains valid for $Q \in (0, L]$. An application of Lemma 4 with $H_1 = 0$, $H_2 = \frac{N}{L}$ yields

$$L^{-11/4}S_0^2 \ll M^2N + (MN^{3/2}X^{1/2})^{4/5}(M^2N^2)^{1/5} + (MN^{7/6}X^{2/3})^{3/4}(M^2N^2)^{1/4} + (MN^{7/4}X^{1/4})^{8/11}(M^2N^2)^{3/11}.$$

This gives the required variant of (3.5) for S_0 and completes the proof.

Lemma 8. *Let α, β, γ be real constants with $\alpha < 1$, $\alpha\beta\gamma \neq 0$. Let (κ, λ) be an exponent pair. Let M, M_1, M_2 be real numbers $\geq \frac{1}{2}$ and $X \geq M_1M_2$. Let*

$$S_2 = \sum_{\substack{m\sim M \\ D\leq mm_1<D'}} \sum_{m_1\sim M_1} \sum_{m_2\sim M_2} a_m b_{m_1,m_2} e\left(\frac{Xm^\alpha m_1^\beta m_2^\gamma}{M^\alpha M_1^\beta M_2^\gamma}\right)$$

where $|a_n| \leq 1$, $|b_{m_1,m_2}| \leq 1$. Then

$$S_2 \ll MM_1M_2(\log 12MM_1M_2)^2\{(M_1M_2)^{-1/2} + \left(\frac{X}{M_1M_2}\right)^{\kappa/(2+2\kappa)} M^{-(1+\kappa-\lambda)/(2+2\kappa)}\}.$$

Proof. By a similar remark to that at the beginning of the last proof, this follows from [1, Theorem 2].

Lemma 9. *Let α, β, γ be real constants with $\alpha(\alpha-1)\beta\gamma \neq 0$. Let H, M, N be real numbers $\geq \frac{1}{2}$, let $X > 0$ and*

$$S = \sum_{h\sim H} \sum_{m\sim M} \max \left| \sum_{n=N_1}^{N_2} e\left(\frac{Xm^\alpha h^\beta n^\gamma}{M^\alpha H^\beta N^\gamma}\right) \right|$$

where the maximum is taken over $1 \leq N_1 \leq N_2 \leq N$. Then

$$S \ll (HNM)^{1+\epsilon}\left(\left(\frac{X}{HMN^2}\right)^{1/4} + \frac{1}{N^{1/2}} + \frac{1}{X}\right).$$

Proof. See [16, Theorem 3].

4. Proof of Theorem

Throughout this section, $k \in \{3, 4, 5\}$. We write

$$y_k = x^{(1-2\theta_k)/(k-1)},$$

so that (1.2) yields

$$R_k(x) = -\sum_{n\leq y_k} \mu(n)\psi(xn^{-k}) + O(x^{\theta_k+\epsilon}).$$

Accordingly it suffices to prove that

$$\sum_{n\sim D} \mu(n)\psi(xn^{-k}) \ll x^{\theta_k+3\epsilon/4} \tag{4.1}$$

for

$$x^{\theta_k} < D < x^{(1-2\theta_k)/(k-1)}. \tag{4.2}$$

((4.1) is trivial for smaller D.)

Let $H \geq 1$. Vaaler [17] (see also [8, Appendix]) gave the approximation

$$\Big|\psi(w) - \sum_{0<|h|\leq H} a_h e(hw)\Big| \leq B(w),$$

where $B(w) = \sum_{|h|\leq H} b_h e(hw)$ is non-negative; the a_h, b_h are given explicitly and satisfy

$$a_h \ll \frac{1}{h} \quad , \quad b_h \ll \frac{1}{H}. \tag{4.3}$$

It follows that

$$\Big|\sum_{n\sim D} \mu(n)\psi(xn^{-k}) - \sum_{0<|h|\leq H} a_h \sum_{n\sim D} \mu(n)e(hxn^{-k})\Big| \leq \sum_{|h|\leq H} b_h \sum_{n\sim D} e(hxn^{-k}).$$

We select $H = [Dx^{-\theta_k}]$. The contribution to the right-hand side from $h = 0$ is

$$\ll DH^{-1} \ll x^{\theta_k}$$

from (4.3). Accordingly, after a splitting-up argument, we need to show that

$$\sum_{h\sim K} c_h \sum_{n\sim D} \mu(n)e(hxn^{-k}) \ll Kx^{\theta_k+2\epsilon/3} \tag{4.4}$$

whenever (4.2) holds, $|c_h| \leq 1$ and

$$\frac{1}{2} \leq K \leq Dx^{-\theta_k}; \tag{4.5}$$

for our proof will show that (4.4) remains valid with 1 in place of $\mu(n)$.

In the rest of this section, we write

$$S_I(D,K,N) = \sum_{h\sim K} c_h \sum_{\substack{m\sim M \\ mn\sim D}} a_m \sum_{n\sim N} e\left(\frac{hx}{(mn^k)}\right),$$

$$S_{II}(D,K,N) = \sum_{h\sim K} \sum_{\substack{m\sim M \\ mn\sim D}} a_m \sum_{n\sim N} b_n e\left(\frac{hx}{(mn^k)}\right),$$

where c_h, a_m, b_n are unspecified numbers of absolute value ≤ 1, and D, K are assumed to satisfy (4.2), (4.5).

Lemma 10. *(i) Suppose that*

$$S_{II}(D,K,N) \ll Kx^{\theta_k+\epsilon/3} \tag{4.6}$$

whenever

$$D^2x^{-2\theta_k} \ll N \ll D^{1/2}. \tag{4.7}$$

Then (4.4) holds.

(ii) Suppose that

$$K < x^{5\theta_k-1}D^{k-7/2}. \tag{4.8}$$

Then (4.4) holds.

(iii) Suppose that

$$KD^{8-k} < x^{10\theta_k-1}. \tag{4.9}$$

Then (4.4) holds.

Proof. (i) Since $\theta_k > 5/(6k+4)$, we deduce from (4.2) that

$$D^2 x^{-2\theta_k} \ll D^{1/3-c}. \tag{4.10}$$

We can thus apply Lemma 3(ii) with the choice

$$D^\lambda = x^{2\theta_k} D^{-1} \quad , \quad D^{1-\lambda} = D^2 x^{-2\theta_k}.$$

Our hypothesis gives (4.6) for the range

$$D^{1-\lambda} \ll N \ll D^{1/2}.$$

We claim that

$$S_I(D, K, N) \ll K x^{\theta_k + \epsilon/3} \tag{4.11}$$

for

$$N \gg x^{1-4\theta_k} D^{3-k}. \tag{4.12}$$

This is a consequence of Lemma 9; we must show that

$$KD\left(\frac{x}{N^{1/4} D^{k+1}}\right)^{1/4} + KDN^{-1/2} + KD(KxD^{-k})^{-1} \ll Kx^{\theta_k}.$$

The first term on the left gives rise to the condition (4.12). The second term produces the condition

$$N \gg D^2 x^{-2\theta_k}. \tag{4.13}$$

Moreover, (4.2) gives

$$D^2 x^{-2\theta_k} < x^{1-4\theta_k} D^{3-k},$$

so that (4.13) follows from (4.12). Finally, it is easy to see that

$$x^{-1} D^{1+k} \ll x^{\theta_k},$$

so that the third term gives no difficulty, proving our claim.

Now

$$D^\lambda > x^{1-4\theta_k} D^{3-k},$$

or equivalently,

$$D^{4-k} < x^{6\theta_k - 1}.$$

For $k = 3$, this follows from $\theta_3 > 3/14$. For $k = 4$, we use $\theta_4 > 1/6$, and for $k = 5$, we require $D > x^{1/10}$, which is a consequence of (4.2). We conclude that (4.11) holds for $N \gg D^\lambda$, and (i) follows at once.

(ii) We apply Lemma 7 to show that (4.6) holds in the range (4.7) under the hypothesis (4.8). Treating the variable h trivially, this reduces to verifying that

$$(KxD^{-k}M^3N^4)^{1/5} + (KxD^{-k}M^{10}N^{11})^{1/16} + (KxD^{-k}M^7N^{10})^{1/11}$$
$$+ MN^{1/2} + (KxD^{-k})^{-1/2} D \ll x^{\theta_k}.$$

The term $MN^{1/2}$ gives rise to the lower bound for N in (4.7). Next,

$$(KxD^{-k})^{-1/2} D \ll D^{k/2+1} x^{-1/2} \ll x^{\theta_k}$$

follows from (4.2), because

$$\theta_k > 3/(4k+2).$$

The condition

$$(KxD^{-k}M^3N^4)^{1/5} \ll x^{\theta_k}$$

may be rewritten

$$KN \ll x^{5\theta_k - 1} D^{k-3}. \tag{4.14}$$

In this form, it follows from (4.8) for $N \ll D^{1/2}$.

The condition

$$(KxD^{-k}M^{10}N^{11})^{1/16} \ll x^{\theta_k}$$

may be rewritten

$$KN \ll x^{16\theta_k - 1} D^{k-10}.$$

Now

$$x^{5\theta_k - 1} D^{k-3} < x^{16\theta_k - 1} D^{k-10}$$

from (4.2), because

$$\theta_k > 7/(11k+3).$$

Finally, the condition

$$(KxD^{-k}M^7N^{10})^{1/11} \ll x^{\theta_k}$$

may be rewritten

$$K^{1/3}N \ll x^{(11\theta_k - 1)/3} D^{(k-7)/3}.$$

To show that this follows from (4.14), it suffices to verify that

$$x^{5\theta_k - 1} D^{k-3} < x^{(11\theta_k - 1)/3} D^{(k-7)/3},$$

and this in turn follows from (4.2). This completes the proof of (ii).

(iii) Similarly, on applying Lemma 6 with $\kappa = \lambda = 1/2$, we must show that (4.7) and (4.9) together imply

$$DN^{-1/2} + DM^{-1/4} + D^{5/4}(KxD^{-k})^{-1/4}N^{-3/4}$$
$$+ D^{5/4}(KxD^{-k})^{-1/4}M^{-3/8} + D^{4/5}(KxD^{-k})^{1/10} + D^{7/8} \ll x^{\theta_k}.$$

To begin with, we observe that (4.9) implies

$$D \ll x^{(10\theta_k - 1)/(8-k)}. \tag{4.15}$$

Thus

$$D^{7/8} \ll x^{(70\theta_k - 7)/(64-8k)} < x^{\theta_k}$$

because

$$\theta_k < 7/(6+8k). \tag{4.16}$$

Since $M \gg D^{1/2}$, it follows also that

$$DM^{-1/4} \ll x^{\theta_k}.$$

As before, the term $DN^{-1/2}$ gives rise to the lower bound on N in (4.7). Next,

$$D^{4/5}(KxD^{-k})^{1/10} = (KD^{8-k})^{1/10} x^{1/10} < x^{\theta_k}$$

from (4.9). Next,

$$D^{5/4}(KxD^{-k})^{-1/4}N^{-3/4} \ll (D^{5+k}x^{-1})^{1/4}(D^2x^{-2\theta_k})^{-3/4}$$
$$= (D^{k-1}x^{-1+6\theta_k})^{1/4} < x^{\theta_k}$$

from (4.2).

Finally, since $M \gg D^{1/2}$,

$$D^{5/4}(KxD^{-k})^{-1/4}M^{-3/8} \ll (D^{5+k}x^{-1})^{1/4}D^{-3/16} = (D^{17+4k})^{1/16}x^{-1/4} \ll x^{\theta_k}.$$

To see this, we use (4.15):

$$D^{17+4k} \ll x^q \quad , \quad \text{where}$$
$$q = (17+4k)(10\theta_k - 1)/(8-k) < 16\theta_k + 4,$$

as a consequence of (4.16). This completes the proof of (iii).

We have yet to use Lemma 8, which is capable of extending the range (4.7) on the left.

Lemma 11. *Suppose that*

$$K^{-1}D^2x^{-2\theta_k} \ll N \ll \min(D^{k-4}x^{6\theta_k-1}, D^{1/2}). \tag{4.17}$$

Then (4.6) holds.

Proof. We apply Lemma 8 with N, K in place of M_1, M_2, and $\kappa = \lambda = 1/2$. The condition $X \gg M_1M_2$ reduces to

$$xD^{-k} \gg N.$$

Since $N \ll D^{1/2}$, this reduces in turn to $\theta_k \geq 3/(4k+2)$, which was used earlier.

Now we need to show that (4.17) implies

$$KD(NK)^{-1/2} + KD(xD^{-k}N^{-1})^{1/6}M^{-1/3} \ll Kx^{\theta_k}.$$

The term $KD(NK)^{-1/2}$ gives rise to the lower bound for N in (4.17). The condition

$$KD(xD^{-k}N^{-1})^{1/6}M^{-1/3} \ll Kx^{\theta_k}$$

is equivalent to

$$D^6\left(\frac{x}{D^{k+2}}\right)N \ll x^{6\theta_k},$$

which reduces to the upper bound for N in (4.17).

Proof of the Theorem. Let D, K satisfy (4.2), (4.5). We must show that (4.4) holds.

Suppose first that $k = 3$. If

$$K < x^{11/74}D^{-1/2}$$

then we are done, by Lemma 10(ii). Suppose now that

$$K \geq x^{11/74}D^{-1/2}. \tag{4.18}$$

We apply Lemma 3(i). Thus it suffices to prove (4.6) for

$$D^{1/4} \ll N \ll D^{2/5} \tag{4.19}$$

and (4.11) for

$$N \gg D^{2/5}. \tag{4.20}$$

Lemma 11 gives (4.6) in the range

$$K^{-1}D^2x^{-17/37} \ll N \ll \min(D^{-1}x^{14/37}, D^{1/2}).$$

From (4.18), (4.2),

$$K^{-1}D^2x^{-17/37} \leq D^{5/2}x^{-45/74} < D^{1/4},$$
$$D^{-1}x^{14/37} > D^{2/5}.$$

Thus (4.6) holds in the required range.

As shown in the proof of Lemma 10, (4.11) holds in the range (4.12), which becomes

$$N \gg x^{3/37}$$

for $k = 3$. Since $x^{3/37} < D^{2/5}$ from (4.2), the discussion of the case $k = 3$ is complete.

Now let $k = 4$. From Lemma 10(ii), we may suppose that

$$K \geq x^{-9/94} D^{1/2}. \tag{4.21}$$

As above, it suffices to prove (4.6) in the range (4.19), and (4.11) in the range (4.20).

Lemma 11 gives (4.6) in the range

$$K^{-1} D^2 x^{-17/47} \ll N \ll x^{4/47}, \tag{4.22}$$

since $D^{1/2} > x^{17/188} > x^{4/47}$. Now

$$K^{-1} D^2 x^{-17/47} < x^{-25/94} D^{3/2} < D^{1/4},$$

since $D < x^{10/47}$ from (4.2). Moreover,

$$x^{4/47} > D^{2/5},$$

also from (4.2). This gives (4.6) in the desired range.

As for (4.11), the range (4.12) becomes

$$N \gg x^{13/47} D^{-1}. \tag{4.23}$$

If $D \geq x^{9/47}$, then we obtain (4.11) in the desired range by combining the ranges (4.22) and (4.23).

It remains to prove (4.4) in the case $D < x^{9/47}$. In this case, (4.5) yields

$$KD^4 \leq D^5 x^{-17/94} < x^{73/94},$$

and (4.4) follows from Lemma 10(iii).

Finally, let $k = 5$. In view of Lemma 10(ii), (iii), we may suppose that

$$K \geq x^{-1/4} D^{3/2} \tag{4.24}$$

and

$$KD^3 \geq x^{1/2}. \tag{4.25}$$

We may apply Lemma 3(iii) with

$$D^a = K^{-1} D^2 x^{-3/10},$$

which was shown in (4.10) to be smaller than $D^{1/3}$. We must show that (4.6) holds for

$$K^{-1} D^2 x^{-3/10} \ll N \ll \max(D^{1/3}, K^{-2} D^4 x^{-3/5}),$$

and that (4.11) holds for

$$N \gg K^{1/2} D^{-1/2} x^{3/20}.$$

We apply Lemma 11. We have to show that

$$\max(D^{1/3}, K^{-2} D^4 x^{-3/5}) \ll D x^{-1/10}$$

(since $Dx^{-1/10} < D^{1/2}$ from (4.2)). The bound

$$D^{1/3} < D x^{-1/10}$$

follows from (4.2). Also,

$$K^{-2} D^4 x^{-3/5} (D x^{-1/10})^{-1} = K^{-2} D^3 x^{-1/2} \leq 1$$

from (4.24). This gives (4.6) in the required range.

As for (4.11), the range (4.12) satisfies our requirements, because we obtain

$$x^{2/5}D^{-2} \leq K^{1/2}D^{-1/2}x^{3/20}$$

on rearranging (4.25). This finishes the discussion for $k = 5$, and completes the proof of the Theorem.

References

[1] R.C. Baker, The square-free divisor problem *Quart. J. Math. Oxford*, 45 (1994), 269-277.

[2] R.C. Baker, Sums of two relatively prime cubes. *Acta Arith.*, 129 (2007), 103-146.

[3] R.C. Baker and G. Kolesnik, On the distribution of p^α modulo one, *J. Reine Angew. Math.*, 356 (1985), 174-193.

[4] R.C. Baker and J. Pintz, The distribution of square-free numbers, *Acta Arith.*, 46 (1985), 73-79.

[5] C.J.A. Evelyn and E.H. Linfoot, On a problem in the additive theory of numbers IV, *Ann. of Math.*, 32 (1931), 261-270.

[6] E. Fouvry and H. Iwaniec, Exponential sums with monomials, *J. Number Theory*, 33 (1989), 311-333.

[7] S.W. Graham, The distribution of square-free numbers, *J. London Math. Soc.*, 24 (1981), 54-64.

[8] S.W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge University Press, 1991.

[9] S.W. Graham and J. Pintz, The distribution of r-free numbers, Acta Math. Hung., 53 (1989), 213-236.

[10] G. Harman, *Prime-detecting Sieves*, Princeton University Press, 2007.

[11] D.R. Heath-Brown, Prime numbers in short intervals and a generalized Vaughan identity, *Canad. J. Math.* 34 (1982), 1365-1377.

[12] D.R. Heath-Brown, The Pjateckii-Sapiro prime number theorem, *J. Number Theory*, 16 (1983), 242-266.

[13] C.-H. Jia, The distribution of square-free numbers, *Sci. China Ser. A* 36 (1993), 154-169.

[14] H.L. Montgomery and R.C. Vaughan, The distribution of square-free numbers, *Recent Progress in Analytic Number Theory (Durham, 1979)*, Vol. 1, Academic Press, 1981, 247-256.

[15] J. Pintz, On the distribution of square-free numbers, *J. London Math. Soc.*, 28 (1983), 401-405.

[16] O. Robert and P. Sargos, Three-dimensional exponential sums with monomials, *J. Reine Angew. Math.*, 591 (2006), 1-20.

[17] J.D. Vaaler, Some extremal problems in Fourier analysis, *Bull. Amer. Math. Soc.*, 12 (1985), 183-216.

[18] J. Wu, On the average number of unitary factors of finite abelian groups, Acta Arith., 84 (1998), 17-29.

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