Math 371, Midterm Exam #2 Study Guide

General information

(1) The exam will cover Chapters 5,6, and 7 (up to 7.3).
(2) Books and notes will not be allowed.
(3) WARNING: this study guide is not meant to be exhaustive. Just because something is not on the study guide does not mean it will not be on the exam.

Basics

(1) You should know everything that was on the first study guide, especially the basic properties of rings, in Chapter 3.
(2) Ring Definitions:
  - a ring, a field, a domain, an integral domain
  - a zero divisor, a unit
  - a ring homomorphism and a ring isomorphism
  - the Cartesian product of two rings
  - a monic polynomial and an irreducible polynomial
  - an ideal
  - the kernel of a homomorphism
  - maximal ideals, prime ideals
  - principal ideals, ideals generated by a finite number of elements
(3) Group Definitions:
  - A group, Abelian groups.
  - Cartesian product of two groups.
  - Order of a group and of an element.
  - Subgroup, center of a group, centralizer of a group.
  - Cyclic group, subgroup generated by a finite number of elements.
(4) Lots of examples of all the things we have discussed, especially:
  - Examples of rings, both commutative and non-commutative, of every order.
  - Examples of subrings and ideals with many different properties (including maximal ideals, non-maximal prime ideals, ideals which are not principal, etc.).
  - A maximal ideal that does not contain all proper ideals in the ring.
  - An infinite ring and an ideal with a finite quotient ring.
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  - A field with 4 elements, and a ring with 4 elements that is not a field.
  - A field F that properly contains the rationals $\mathbb{Q}$ and is properly contained in the reals $\mathbb{R}$ (i.e., $\mathbb{Q} \subset F \subset \mathbb{R}$).
  - Non-Abelian groups: S_n, D_n, $GL_n(F)$, $SL_n(R)$.
  - Abelian groups: $\mathbb{Z}$, $\mathbb{Z}_n$, U_n.
  - An element of finite order contained in a group of infinite order.
  - Cyclic groups of all orders—both infinite and finite.
  - Groups which are not cyclic, including a (sub)group generated by two elements which is not cyclic.
  - A group with a non-trivial center.

Theorems you should know and be able to state and prove and use

- The First Isomorphism Theorem for rings.
- If R is a commutative ring with identity and I is an ideal of R, then R/I is an integral domain if and only if I is a prime ideal.
- The center of a group is a subgroup.

Theorems you should be able to use

- The simple criterion for checking that a subset is an ideal (Thm 6.1) or that a subset is a subgroup (Thm 7.10).
- If R is a commutative ring with identity and I is an ideal of R, then R/I is a field if and only if I is a maximal ideal.
- The set of cosets of an ideal forms a ring (the quotient ring). Specifically, addition and multiplication of cosets of an ideal are well defined.
- The kernel of a homomorphism is an ideal.
- for every ring R and every ideal I in R, there is a natural surjective homomorphism R to R/I, given by $r \mapsto r + I$ (Theorem 6.12).
- In a commutative ring with identity, every maximal ideal is prime.
- Every ring is an Abelian group under addition.
- If R is a ring with identity, then the set of units of R is a group under multiplication.
- The identity element of a group is unique.
- Cancellation holds in a group; that is, $ab = ac$ implies that $b = c$.
- In a group, inverses are unique.
- Every subgroup of a cyclic group is cyclic.

Sample problems

(1) Prove that the set $\{a + b\sqrt{3} | a, b \in \mathbb{Q}\}$ is a field and is isomorphic to $\mathbb{Q}[x]/(x^2 - 3)$.
(2) Explain why multiplication of cosets in R/J makes sense only if J is an ideal.
(3) Construct a field of order 4.
(4) Prove that $\mathbb{Z}_4$ is not a field.
(5) Give an example of a maximal ideal in a ring that does not contain all proper ideals of the ring.
(6) Give an example of a prime ideal I in $\mathbb{Z} \times \mathbb{Z}$ that is not maximal. Describe the quotient ring $\mathbb{Z} \times \mathbb{Z}/I$.
(7) Prove that $H := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a = \pm 1, b \in \mathbb{Z} \right\}$ is a subgroup of $GL(2, \mathbb{Q})$.
(8) Show that the group $\mathbb{Z}_5 \times \mathbb{Z}_2$ is cyclic but that $\mathbb{Z}_6 \times \mathbb{Z}_2$ is not cyclic.
(9) Prove that the identity is unique in a group.
(10) Prove that inverses are unique in a group.
(11) Give an example of a non-Abelian group of order 24.
(12) Are there any groups of order 3 which are not cyclic? If so, give an example, if not, prove it.
(13) Let T be the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let I be the set $\{g \in T : g(-2) = 0\}$. Prove that I is an ideal and that $T/I \cong \mathbb{R}$.