Miscellaneous worked problems:

Section 7.8, problem 19: Let $f : G \to H$ be a surjective homomorphism of groups with kernel K.

Note that if $f(x) = f(y)$, then $f(xy^{-1}) = f(x)f(y)^{-1} = e$, so $xy^{-1} \in K$. So if P is a subgroup of G, and P contains K, then $xy^{-1} \in P$. Thus (assuming $f(x) = f(y)$) either x and y are both in P, or neither is in P.

Now if P is a subgroup of G, containing K, let $\psi(P) = \{f(x) \big| x \in P\}$. (This might be denoted $f(P)$, but I want to emphasize that ψ will be a map of subgroups.) By Theorem 7.19 (3), $\psi(P)$ is a subgroup of H, as desired. We shall prove that ψ is bijective (from the family of all subgroups P containing K to the family of all subgroups of H).

Suppose P and Q are *distinct* subgroups of G, both containing K, then there exists an element of G which is in one of the subgroups but not in the other. Assume w.l.o.g. that $x \in P \setminus Q$. From our discussion above, any elements y having the same f-image as x will also lie in P and not in Q. Thus $f(x)$ is in $\psi(P)$ and not in $\psi(Q)$, proving that $\psi(P) \neq \psi(Q)$. Thus ψ is injective.

Let U be any subgroup of H, and let $R = f^{-1}(U) = \{x \in G \big| f(x) \in U\}$. In problem 23 of 7.4, we proved that $f^{-1}(U)$ is a subgroup of G. Since $e \in U$, if follows that $f^{-1}(U)$ contains K, the kernel of f. Thus R is a subgroup of G containing K. Since f is surjective, it follows that $f\big(f^{-1}(U)\big) = U$, so $\psi(R) = U$, proving that ψ is surjective. $\square$

Section 7.8, problem 24: Let K and N be subgroups of a group G, with $N \triangleleft G$. By Exercise 18 of 7.6 (you should review that problem and solution) the set NK, defined by $\{nk \big| n \in N, k \in K\}$ is a subgroup of G, and it contains both N and K as subgroups.

(a) If $x \in NK$ then $x \in G$, so $xNx^{-1} = N$ (because N is normal in G). Therefore, we've shown that N is normal in NK.

(b) Note that if $k \in K$, then Nk is a *coset* of N in NK. In other words it's an *element* of NK/N. So the map $f(k) = Nk$ is a function from K to NK/N. We must show it's a surjective homomorphism with appropriate kernel.

If Nx is an arbitrary element of NK/N, then x must be an element of NK, so $x = nk$ for some $n \in N$ and $k \in K$. Thus $Nx = Nnk = Nk = f(k)$, proving the map is onto. It's a homomorphism because $f(k_1k_2) = N(k_1k_2) = (Nk_1)(Nk_2) = f(k_1)f(k_2)$.

The kernel of f is the set of $k \in K$ with $f(k)$ equal to the identity in NK/N, which is the coset N. Thus the kernel is $\{k \in K \big| Nk = N\} = \{k \in K \big| k \in N\} = K \cap N$, as desired.

(c) By the First Isomorphism Theorem for groups, $K/(K \cap N) \cong NK/N$.