ULTRACONNECTED AND CRITICAL GRAPHS

by

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ABSTRACT

ULTRACONNECTED AND CRITICAL GRAPHS

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We investigate the ultraconnectivity condition on graphs, and provide further connections between critical and ultraconnected graphs in the positive definite partial matrix completion problem. We completely characterize when the join of graphs is ultraconnected, and prove that ultraconnectivity is preserved by Cartesian products. We completely characterize when adding a vertex to an ultraconnected graph preserves ultraconnectivity. We also derive bounds on the number of vertices which guarantee ultraconnectivity of certain classes of regular graphs. We give results from our exhaustive enumeration of ultraconnected graphs up to 11 vertices. Using techniques involving the Lovász theta parameter for graphs, we prove certain classes of graphs are critical (and hence ultraconnected) in the positive definite partial matrix completion problem.

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1 Introduction

Ultraconnected graphs arise in connection with various partial matrix completion problems: Given a matrix in which only some of the entries are known, is it possible to complete the matrix to have some specified property? A thorough understanding of a particular matrix completion problem lends itself to understanding how the structure of a matrix affects the property.

In the introductory discussion below, we will assume familiarity with some terminology. Please refer to Section 2 for precise definitions of terms.

At least two widely studied completion problems are linked to the study of ultraconnected graphs: the positive (semi-)definite completion problem, and the Euclidean distance completion problem. The former is useful in probability and statistics, image enhancement, systems engineering, and geophysics. The latter is useful in determining the atomic configuration of molecules.

The positive semidefinite and the positive definite completion problems are equivalent [see 6, Proposition 2]. Throughout the remainder of the thesis, we will work with the positive definite completion problem, with the understanding that equivalent statements apply in the positive semidefinite completion problem.

Throughout this thesis, unless otherwise stated, we will assume that partial matrices are partial real symmetric matrices with the diagonals specified. In addition, we will assume the partial matrices are partial positive definite.

In 1984, Grone, Johnson, Sá, and Wolkowicz [6] considered the question of completing an n-by-n Hermitian partial positive definite matrix with diagonals specified to a positive definite matrix. We restrict the statement of their result to the real case. Grone et al. associated a graph with the pattern of specified entries, which had the vertex set $\{1, 2, \ldots, n\}$ and edge set $\{(i, j) \mid a_{ij} \text{ is specified}\}$. They determined that the partial matrix was guaranteed to have a positive definite completion, regardless of the values of the specified entries, if and only if the associated graph was chordal. The connection between a graph-theoretic condition and a matrix completion problem sparked a wide study of completion problems in terms of graph conditions. See Hogben [10] for a survey of these graph-theoretic methods in various completion problems.

The results of Grone et al. [6] affirmed the existence of a positive definite completion for any partial matrix associated with a chordal graph, and the existence of a set of specified entries for a partial matrix associated with a non-chordal graph that made the matrix non-completable. However, the existence of a positive definite completion for a partial matrix associated with a non-chordal graph is still not completely understood. In other words, when considering a real symmetric partial positive definite matrix with diagonals specified which has an associated non-chordal graph, what conditions are necessary on the specified entries to guarantee a positive definite completion? This question has been explored not only for the positive definite completion problem, but also for the closely related Euclidean distance matrix completion problem, and various graph-theoretic conditions have been proposed and studied [see 1, 2, 13, 15].

Understanding the real symmetric positive definite completion problem for all matrices reduces to understanding it for those matrices associated with a limited class of graphs called critical graphs. A critical graph is one that is associated with some real symmetric partial positive definite matrix that is not positive definite completable, but every proper principal submatrix is positive definite completable. In other words, a partial matrix associated with a non-critical graph has the property that if each proper principal submatrix is completable, then the whole partial matrix is completable. Thus, given a certain partial matrix, we need only to examine the associated graph for induced critical subgraphs. If each critical subgraph corresponds to a completable principal submatrix of the original partial matrix, then the whole partial matrix is completable. For further details, see Barrett et al. [2].

A major result in Barrett, Johnson, and Loewy [2] is that every critical graph satisfies a new graph-theoretic connectivity condition, which they call ultraconnectivity. A graph G is ultraconnected if, for any subset S of the vertices, the number of components in $G-S$ minus the number of edges missing in $G[S]$ is less than or equal to 1. Up to 6 vertices, a graph is critical if and only if it is ultraconnected. Barrett et al. conjecture that this is always the case.

The value of understanding ultraconnected graphs extends beyond the real symmetric positive definite completion problem. To see this, first broaden our class of partial matrices to all those that are combinatorially symmetric (i.e., a_{ij} is defined iff a_{ji} is defined, but a_{ij} does not necessarily equal a_{ji}), and consider a property P that is defined for all such completed matrices. Then as noted by Barrett et al. [2], the completion problem involving the property P has ultraconnected critical graphs (where critical matrices are partial matrices that have P completable principal submatrices, but are not themselves P completable) if P satisfies the following three conditions:

P **is inherited:** A matrix satisfying P implies that all principal submatrices satisfy P.

P **is chordal:** There is a chordal existence theorem for the completion problem, similar to the one for the positive definite completion problem mentioned ealier from Grone et al. [6].

P **is implicitly convex:** Let $D = \{t \mid t \text{ is an entry of some matrix with property } P\}$. Then there is a function f, $1-1$ on D, such that for each positive integer n, the set $\{(f(a_{ij})) \mid A = (a_{ij}) \text{ is an } n\text{-by-}n \text{ matrix with property } P\}$ is convex.

The property of being an Euclidean distance matrix satisfies these criteria. Therefore the critical graphs in the Euclidean distance completion problem are ultraconnected.

In this thesis, we investigate the ultraconnectivity and criticality condition on graphs. In Section 2, we develop the terminology used throughout the rest of the thesis. In Section 3, we completely characterize when the join of graphs is ultraconnected. In Section 4, we completely characterize when adding a vertex to an ultraconnected graph preserves ultraconnectivity. Section 5 shows that Cartesian products preserve ultraconnectivity. In Section 6, we turn our attention to classes of regular graphs, and derive bounds on the number of vertices which guarantee ultraconnectivity of graphs. Section 7 gives results from our exhaustive enumeration of ultraconnected graphs up to 11 vertices. In Section 8, we use new techniques involving the Lovász theta parameter for graphs to prove classes of graphs are critical in the positive definite completion problem. Finally, in Section 9, we list open questions and further directions for study uncovered during our research.

We have also included in Appendix A a listing of code we have used to determine ultraconnectivity of graphs.

2 Terminology

We start by giving precise definitions for terminology we use throughout the thesis involving graphs and matrices. Refer to West [22] for other definitions and further details on topics in graph theory. The reader that is familiar with graph theory terminology may safely skip sections 2.1 through 2.3. Refer to Horn and Johnson [11] for further details and definitions of matrix-related topics.

2.1 Graphs

Definition 1. A *graph* G is an ordered pair $(V(G), E(G))$ of a nonempty vertex set $V(G)$ and an *edge* set $E(G)$ consisting of distinct unordered pairs of distinct elements of $V(G)$. We write an edge $\{x, y\}$ as xy for convenience. The graph G is more precisely called a *simple graph*.

Definition 2. An edge $e = uv \in E$ is said to *join* the vertices u and v, and u and v are referred to as the *ends* of e. Two vertices that are joined by an edge are *adjacent*. Each end of an edge is said to be *incident* with the edge. Two edges that are incident with the same vertex are said to be *adjacent*.

Definition 3. A *complete graph*, or a *clique*, is a graph in which the vertices are all pairwise adjacent. The complete graph with n vertices is written K_n. On occasion, we will use the term *clique* to denote the set of vertices of a complete graph.

Definition 4. A *bipartite graph* G is a graph in which $V(G)$ can be partitioned into two subsets X and Y such that each edge has one end in X and one end in Y. The partition (X, Y) is called a *bipartition* of the graph. A *complete bipartite graph* is a bipartite graph for which there exists a partition (X, Y) such that each vertex in X is adjacent to every vertex in Y. If $|X| = n$ and $|Y| = m$, then the complete bipartite graph is denoted $K_{m,n}$.

Definition 5. Let $G = (V(G), E(G))$ and $G' = (V(G'), E(G'))$ be graphs. The graphs G and G' are *isomomorphic* if there exists a bijective function $\varphi\colon V(G) \to V(G')$ such that $xy \in E(G)$ iff $\varphi(x)\varphi(y) \in E(G')$. The function φ is an *isomorphism*. If G is identical to G', then φ is an automorphism. We will not distinguish between isomorphic graphs.

Definition 6. Let $G = (V(G), E(G))$ and $G' = (V(G'), E(G'))$ be graphs. Then G' is a *subgraph* of G, written $G' \subseteq G$, if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. If $G' \subseteq G$ and G' contains all possible edges from G (i.e., $E(G') = \{xy \mid x, y \in V(G'),\, xy \in E(G)\}$), then G' is an *induced subgraph* of G. In this case, we say that $V(G')$ induces G' in G, and write G' as $G[V(G')]$.

Definition 7. Let G be a graph, $v_i \in V(G)$, and $x_j \in E(G)$. Then the *removal of a vertex* v_i is $G - v_i = G[V \setminus \{v_i\}]$. The *removal of an edge* x_j is $G - x_j = (V, E \setminus \{x_j\})$. Similarly, $G - \{v_1, \ldots v_n\}$ is defined as the sequential removal of each v_i, and $G - \{x_1, \ldots x_n\}$ is defined as the sequential removal of each x_j. If $v_1, v_2 \in V(G)$ and $v_1v_2 \notin E(G)$, then the *addition of an edge* v_1v_2, denoted $G + v_1v_2$, is the graph with vertex set $V(G)$ and edge set $E(G) \cup \{v_1v_2\}$.

Definition 8. Let G be a graph and $v \in V$. The *degree* $\deg(v)$ of v in G is the number of edges incident with v in G. The minimum degree among the vertices of G is denoted $\delta(G)$, while the maximum is denoted $\Delta(G)$.

Definition 9. If $\delta(G) = \Delta(G)$, then we say G is a *regular* graph of degree $r = \delta(G)$.

Definition 10. Let G be a graph. The *clique number* $\omega(G)$ is the number of vertices in the largest clique that is a subgraph of G. The *independence number* $\alpha(G)$ of G is $\alpha(G) = \omega(G^c)$, the size of the largest clique in G^c (see Definition 11 for a definition of G^c).

2.2 Operations on Graphs

Several operations are commonly used with graphs.

Definition 11. The *complement* G^c of a graph G is the graph having vertex set $V(G^c) = V(G)$ and edge set $E(G^c) = \{xy \mid x, y \in V(G),\, xy \notin E(G)\}$.

Definition 12. The *union* G of two graphs G_1 and G_2 having distinct vertex sets, denoted $G = G_1 \cup G_2$, has the vertex set $V(G) = V(G_1) \cup V(G_2)$ and the edge set $E(G) = E(G_1) \cup E(G_2)$

Definition 13. Let G_1 and G_2 be graphs having distinct vertex sets. The *join* G of G_1 and G_2, denoted $G = G_1 \vee G_2$, has the vertex set $V(G) = V(G_1) \cup V(G_2)$ and the edge set

$$E(G) = E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1),\, y \in V(G_2)\}.$$

The join of two graphs is their union with all possible edges between the two graphs.

Remark 1. Note that $(G_1 \cup G_2)^c = G_1^c \vee G_2^c$ for two graphs G_1 and G_2.

Remark 2. In terms of complements, the complete bipartite graph can be expressed as $K_{m,n} = K_m^c \vee K_n^c$. We generalize this to define the *complete multipartite graph* as $K_{n_1,n_2,\ldots,n_p} = K_{n_1}^c \vee K_{n_2}^c \vee \cdots \vee K_{n_p}^c$.

Definition 14. Let G_1 and G_2 be graphs having distinct vertex sets. The *Cartesian product* $G = G_1 \times G_2$ of G_1 and G_2 is the graph having vertex set $V(G) = \{(x, y) \mid x \in V(G_1), y \in V(G_2)\}$ and edge set specified by putting (x, y) adjacent to (x', y') iff (1) $x = x'$ and $yy' \in E(G_2)$, or (2) $y = y'$ and $xx' \in E(G_1)$.

One way to think of the Cartesian product $G_1 \times G_2$ is to imagine a copy of G_1 placed at every vertex of G_2, with corresponding vertices of different copies of G_1 joined when there is an edge in G_2.

Remark 3. The Cartesian product of a graph with K_1 is itself, i.e., $G \times K_1 = G$.

2.3 Connectivity

Definition 15. A *walk* of a graph G is a finite nonempty alternating sequence of vertices and edges $v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n$, beginning and ending with vertices, such that for $1 \leq i \leq n$, the ends of e_i are v_{i-1} and v_i. We call the walk a v_0v_n walk. Because there is a unique edge between adjacent vertices, the walk is also expressed as $v_0v_1 \ldots v_n$, the edges being inferred from the adjacent vertices.

The walk is *closed* if $v_0 = v_n$, and is *open* otherwise. If the edges $e_1 \ldots e_n$ are distinct, the walk is a *trail*. If the vertices $v_0 \ldots v_n$ are distinct, the walk is a path (and also necessarily a trail). The length of a walk $v_0 \ldots v_n$ is n, the number of edges in it.

Definition 16. A nontrivial closed trail is a *circuit*. A circuit $v_1 \ldots v_nv_1$, $n \geq 3$, in which the first n vertices are distinct is the *cycle* C_n of length n. A cycle is *even* or *odd* if the length of the cycle is respectively even or odd.

Definition 17. A graph G is *chordal* if C_k, $k \geq 4$, is not an induced subgraph of G.

Definition 18. Let $S \subseteq V(G)$. If G is connected, but $G - S$ is not, then S is a *separator* of G. If $S = \{v\}$ contains only one vertex, then v is a *vertex separator*. If S is a clique, then S is a *clique separator*.

Definition 19. A graph is *connected* if every pair of vertices is joined by a path. A maximal connected subgraph of a graph G is called a *connected component* or *component* of G.

Remark 4. The graph $G_1 \times G_2$ is connected iff both G_1 and G_2 are connected [see 12, p. 28].

2.4 Ultraconnectivity

Definition 20. Let G be a graph. Let $S \subseteq V(G)$. Define $c(S, G)$ to be the number of components in the graph $G - S$, and $d(S, G)$ to be the number of edges missing in $G[S]$ (i.e., the number of edges in $(G[S])^c$). If S is not specified, then it is assumed that $S = V(G)$, i.e., $d(G) = d(V(G), G)$.

Definition 21. The *ultraconnected degree* $\gamma(G)$ of a graph G is defined as

$$\gamma(G) = \max_{S \subseteq V(G)} (c(S,G) - d(S,G)). \tag{1}$$

The graph G is *ultraconnected* iff $\gamma(G) \leq 1$.

Remark 5. Let G be an ultraconnected graph. Then $d(\emptyset, G) = 0$, so $c(\emptyset, G) \leq 1$. In other words, G has one component, and so G is connected. Now let S be the vertices of any clique in G. Then $d(S,G) = 0$, so $c(S,G) \leq 1$, i.e., $G - S$ has one component. This implies that G does not have a clique separator. In particular, G does not have a vertex separator.

Remark 6. Note that if $S \subseteq S'$, then $d(S,G) \leq d(S',G)$. Also $c(V(G),G) = 0$ and $d(V(G),G) = d(G)$, but $S = V(G) \setminus \{v\}$ for any $v \in V(G)$ produces $c(S,G) = 1$ and $d(S,G) \leq d(G)$. So $c(S,G) - d(S,G) > c(V(G),G) - d(V(G),G)$ for some $S \neq V(G)$. Therefore, $\gamma(G)$ is never found by taking $S = V(G)$.

Remark 7. We see that $\gamma(K_n) = 1$, since when taking any proper subset $S \subseteq V(K_n)$, $d(S, K_n) = 0$ and $c(S, K_n) = 1$. Also, $\gamma(K_n^c) = n$, since for any subset $S \subseteq S' \subseteq V(K_n^c)$, we have $c(S, K_n^c) \geq c(S', K_n^c)$, and $d(S, K_n^c) \leq d(S', K_n^c)$. Therefore taking $S = \emptyset$ gives us the maximum $c(\emptyset, K_n^c) - d(\emptyset, K_n^c) = n$.

2.5 Matrices

Let A be an n-by-n matrix. Let A_i be the submatrix of A formed by removing the ith column and row.

The eigenvalues of A will be denoted $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$, so that $\lambda_1(A)$ is the largest eigenvalue of A.

Definition 22. An n-by-n matrix A is *positive semidefinite* if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$, or equivalently, if $\lambda_n(A) \geq 0$. The matrix A is *positive definite* if $x^T A x > 0$ for all $x \in \mathbb{R}^n$, or equivalently, if $\lambda_n(A) > 0$.

2.6 Partial Matrices

Definition 23. A matrix A is a *partial matrix* if some values are specified, while the remaining unspecified values are variables with a certain domain. A *partial real* matrix is one in which the specified values and the domain for the unspecified values are all real numbers. A *partial symmetric* matrix is one having the property that if a_{ij} is specified, then a_{ji} is specified and $a_{ij} = a_{ji}$. A partial matrix is *partial positive definite* if all fully specified principal submatrices are positive definite.

Remark 8. In this thesis, unless otherwise noted, we assume partial matrices have all three of these properties, i.e., partial real, partial symmetric, and partial positive definite. We also assume that the diagonal entries of a partial matrix are always specified.

Definition 24. For a given property P, a P *completion* of a partial matrix A is a set of values for the unspecified entries of A such that the fully specified matrix has the property P. In this case, A is said to be P *completable*. If P is understood, then we just say A has a completion or is *completable*.

2.7 Critical Matrices and Critical Graphs

Definition 25. In the P completion problem, an n-by-n partial matrix A is *critical* if every proper principal submatrix is P completable, but A is not. In this case, the set of specified entries of A constitute *critical data* for A.

Remark 9. Since positive definiteness is an inherited property, we can simplify the above statement. In the positive definite completion problem, an n-by-n partial matrix A is critical if every $(n-1)$-by-$(n-1)$ principal submatrix is positive definite completable, but A is not.

Definition 26. An n vertex undirected simple graph can be associated with every partial symmetric matrix, where an edge ij, $i \neq j$, appears in the graph if the ij entry of the matrix is specified. A *critical graph* is a graph for which an associated critical matrix exists. Critical data for the graph is a set of real labels for the edges, corresponding to entries in the specified positions of the associated matrix, which make the partial matrix critical.

Remark 10. A major result of Barrett et al. is that every critical graph is ultraconnected [2, Theorem 4.1]. In the same paper, Barrett et al. also conjectured that every ultraconnected graph is critical.

3 Ultraconnectivity and Joins

Two qualities of a graph determine the value of γ: the connectivity of the graph, and the edges missing in the graph. If a graph is very well connected, then the number of components in $G - S$ will generally be small, and $\gamma(G)$ will tend to be small. On the other hand, if a graph has many edges missing, then the number of edges missing in $G[S]$ will generally be large, and again $\gamma(G)$ will tend to be small.

When several graphs are joined together, every possible edge is added between the graphs. This tends to increase the connectivity of the resulting graph greatly, which in turn tends to decrease the value of γ for the resulting graph.

If G_i are graphs having associated matrices A_i, then $\bigvee G_i$ has an associated matrix of the form

$$\begin{bmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ * & & & A_n \end{bmatrix},$$

where the "*" represents specified entries.

The following theorem completely characterizes when the join of graphs is ultraconnected.

Theorem 1. *Let $G_1, G_2, \ldots, G_n$ be graphs, and let $G = \bigvee_{i=1}^{n} G_i$ be their join. Then $\gamma(G) \leq 1$ (i.e., G is ultraconnected) if and only if $\gamma(G_k) \leq 1 + \sum_{i \neq k} d(G_i)$ for all $k = 1, \ldots, n$.*

Proof. Let S be a subset of $V(G)$. If $G - S$ contains vertices from two different G_i's, then $c(S, G) = 1$, and therefore $c(S, G) - d(S, G) \leq 1$.

Assume $G - S$ contains vertices of only G_k (i.e., $V(G - S) \subseteq V(G_k)$). Then the components in $G - S$ are just the components of $G_k - (S \cap V(G_k))$, so $c(S, G) = c(S \cap V(G_k), G_k)$. The number of edges missing in $G[S]$ is the sum of the edges missing in each G_i, $i \neq k$, plus the number of edges missing in $G_k[S \cap V(G_k)]$, so

$$\begin{aligned} d(S, G) &= d(S \cap V(G_k), G_k) + \sum_{i \neq k} d(S \cap V(G_i), G_i) \\ &= d(S \cap V(G_k), G_k) + \sum_{i \neq k} d(G_i). \end{aligned}$$

Thus,

$$c(S, G) - d(S, G) = c(S \cap V(G_k), G_k) - d(S \cap V(G_k), G_k) - \sum_{i \neq k} d(G_i) \qquad (2)$$

By definition of $\gamma(G_k)$, equation (2) attains a maximum when $c(S \cap V(G_k), G_k) - d(S \cap V(G_k), G_k) = \gamma(G_k)$. Thus, the graph is ultraconnected if and only if

$$\gamma(G_k) - \sum_{i \neq k} d(G_i) \leq 1, \qquad (3)$$

and the result follows. □

Many corollaries immediately follow from the above result.

Corollary 2. *If G_1 and G_2 are ultraconnected graphs, then $G_1 \vee G_2$ is ultraconnected.*

Proof. Since $\gamma(G_k) \leq 1$, the condition is always met. □

Corollary 3. *Let G be a graph. The graph $G \vee K_n$ is ultraconnected if and only if G is ultraconnected.*

Proof. The reverse implication follows immediately from Corollary 2, noting that K_n is ultraconnected.

Assume $G \vee K_n$ is ultraconnected. Then since $d(K_n) = 0$, we have $\gamma(G) \leq 1$, and thus G is ultraconnected. □

Corollary 4. *Let $G = K_{n_1, n_2, \ldots, n_p} = K_{n_1}^c \vee K_{n_2}^c \vee \cdots \vee K_{n_p}^c$, where $n_p \geq n_i$ for $i < p$. Then G is ultraconnected if and only if $n_p \leq 1 + \sum_{i<p} \binom{n_i}{2}$.*

Proof. Recall from Remark 7 (page 6) that $\gamma(K_n^c) = n$, and note that $d(K_n^c) = \binom{n}{2}$.

($\Longrightarrow$). Assume that G is ultraconnected. Then we have from Theorem 1

$$n_p \leq 1 + \sum_{i<p} \binom{n_i}{2}.$$

($\Longleftarrow$). Assume that $n_p \leq 1 + \sum_{i<p} \binom{n_i}{2} = 1 + \sum_{i \neq p} d(K_{n_i}^c)$. Observe that for all $i \neq p$, $\gamma(K_{n_i}^c) = n_i \leq 1 + \binom{n_p}{2} = 1 + d(K_{n_p}^c)$ because $n_p \geq n_i$. Therefore, by Theorem 1, G is ultraconnected. □

Observation 5. Let $G = K_{1,1,\ldots,1,n}$. Then G is ultraconnected if and only if $n = 1$.

This observation follows both from Corollary 3 (since $K_n = K_{1,1,1,\ldots,1}$ and K_n^c is ultraconnected iff $n = 1$) and Corollary 4.

Corollary 6. *The graph $K_{m,n}$, $m \leq n$, is ultraconnected if and only if $m \leq n \leq \frac{1}{2}(m^2 - m + 2)$.*

Proof. Since $1 + \binom{m}{2} = \frac{1}{2}(m^2 - m + 2)$, the result follows from Corollary 4. □

Corollary 7. *Let $G = K_{n_1,n_2,\ldots,n_p}$, with $n_i = n > 1$ for $i = 1, \ldots, p$. Then G is ultraconnected if and only if $p \geq 2$.*

Proof. From Corollary 4, G is ultraconnected iff

$$\begin{aligned} n_p \leq 1 + \sum_{i<p} \binom{n_i}{2} &= \frac{1}{2}\left(\sum_{i<p} n_i^2 - \sum_{i<p} n_i + 2\right) \\ \Longleftrightarrow\ 2n &\leq (p-1)n^2 - (p-1)n + 2 \\ \Longleftrightarrow\ 2(n-1) &\leq (p-1)n^2 - (p-1)n \\ \Longleftrightarrow\ 2(n-1) &\leq p(n^2 - n) - (n^2 - n) \\ \Longleftrightarrow\ 2(n-1) + n^2 - n &\leq p(n^2 - n) \\ \Longleftrightarrow\ \frac{2}{n} + 1 &\leq p. \end{aligned}$$

The result then follows. □

Corollary 8. *Let G be a complete graph minus the edges of m disjoint cliques, each of size $q > 1$. Then G is ultraconnected if and only if $m \geq 2$.*

Proof. The graph G can be written as $G = K_{1,1,\ldots,1,q,q,\ldots,q} = K_r \vee K_{q,q,\ldots,q}$, where there are m q's and r 1's. From Corollary 3, G is ultraconnected iff $K_{q,q,\ldots,q}$ is ultraconnected. From Corollary 7, this is true iff $m \geq 2$. □

In particular, taking p to be 2 and 3, a complete graph minus nonadjacent edges or edges of disjoint triangles is ultraconnected if and only if more than one is removed.

4 Ultraconnectivity and Vertex Addition

One possible way of getting ultraconnected graphs is to build them up inductively from graphs having one less vertex. To do this, we must understand when adding a vertex to a graph produces an ultraconnected graph. Corollary 3 characterizes when the resulting graph is ultraconnected if the new vertex is joined to all other vertices of the original graph. In this section, we completely characterize when adding a vertex to an ultraconnected graph produces an ultraconnected graph.

First, we state a generalization of Corollary 3. The statement of Theorem 9 is due to Nephi Noble in 1998, but the proof given is original.

Theorem 9. *Let G be an ultraconnected graph. Let H be a graph obtained by adjoining a vertex v to all vertices of G except one, u. Then H is ultraconnected if and only if G is not complete.*

Proof. First, suppose that G is complete. Then H is a complete graph minus one edge. Thus, from Corollary 8, H is not ultraconnected.

Now assume that G is not complete. Let $S \subseteq V(H)$. Notice that if $v \in S$, then the number of components in $H - S$ is the same as in $G - (S \setminus \{v\})$. There are four cases to consider.

$v \in S$, $u \in S$: Then $H[S]$ has only one more missing edge than $G[S \setminus \{v\}]$. Therefore,

$$c(S, H) - d(S, H) = c(S \setminus \{v\}, G) - (d(S \setminus \{v\}, G) + 1) \leq 1 - 1 = 0.$$

$v \in S$, $u \notin S$: Then $H[S]$ has the same number of missing edges as $G[S \setminus \{v\}]$, so

$$c(S, H) - d(S, H) = c(S \setminus \{v\}, G) - d(S \setminus \{v\}, G) \leq 1.$$

$v \notin S$, $u \in S$: Since v is adjacent to all the vertices in $H - S$, we have $c(S, H) = 1$, so $c(S, H) - d(S, H) \leq 1$.

$v \notin S$, $u \notin S$: Note that since $v \notin S$, $d(S, H) = d(S, G)$. Observe that since v is adjacent to all vertices in $H - S$ except u, $c(S, H) \leq 2$. If $c(S, H) \leq 1$, then $c(S, H) - d(S, H) \leq 1$, so assume that $c(S, H) = 2$. Then there are two possibilities: either $\{u, v\}$ is a proper subset of $V(H) \setminus S$ or not.

$\{u, v\} \subsetneq V(H) \setminus S$: Since $c(S, H) = 2$, u is not adjacent to any other vertex in $H - S$, and thus is not adjacent to any other vertex in $G - S$. Thus $c(S, G) \geq 2$. Since G is ultraconnected, $c(S, G) - d(S, G) \leq 1$. Therefore, $d(S, H) = d(S, G) \geq c(S, G) - 1 \geq 1$. Thus, $c(S, H) - d(S, H) \leq 2 - 1 = 1$.

$\{u, v\} = V(H) \setminus S$: Suppose that $d(S, G) = 0$, i.e., $G[S]$ is complete. Because G is not complete, there exists a vertex $w \in S$ such that $uw \notin E(G)$. Also, since G is connected, u is connected to some vertex in S. The vertices in S adjacent to u form a clique separator of u and w in G. Since G is ultraconnected, this is a contradiction. Thus, $G[S]$ is not a clique, and $d(S, H) = d(S, G) \geq 1$. Therefore, $c(S, H) - d(S, H) \leq 2 - 1 = 1$.

Since each of these cases gives $c(S,H) - d(S,H) \leq 1$, the graph H is ultraconnected. $\square$

Using Theorem 9, we can construct some of the ultraconnected graphs in Barrett et al. [2]. In particular, we can construct $\hat{W}_4$ from C_4 and graph 47 from C_5. We can also construct the 6 vertex ultraconnected graphs 8 and 9 from W_5, and graphs 14, 17, and 18 from $\hat{W}_4$ [see 2, p. 124].

Notice that for an ultraconnected graph with n vertices to be constructed by Theorem 9, the graph must have a vertex of degree $n-2$. Not all of the 6 vertex ultraconnected graphs can be constructed using this theorem because several do not have a vertex of degree 4. Several of those that have a degree 4 vertex still cannot be constructed this way because they do not have the required ultraconnected induced subgraph.

We can generalize Theorem 9 to the case of adjoining a vertex to fewer vertices of an ultraconnected graph.

Theorem 10. *Let G be an ultraconnected graph. Let H be a graph obtained by adjoining a vertex v to all vertices of G except those in $T \subseteq V(G)$. Then H is ultraconnected if and only if for every subset $S \subseteq V(G)$, the number of components of $G - S$ consisting entirely of elements of T is less than or equal to $d(S,G)$.*

Proof. ($\Longrightarrow$). Suppose that there is a subset $S \subseteq V(G)$ such that the number of components of $G - S$ consisting entirely of elements of T is greater than $d(S,G)$. Let m be the number of components of $G - S$ containing vertices only in T. Then $c(S,H) = m+1$, since $v \notin S$, and paths through v will join all components of $H - S$ that contain vertices outside of T. Also, $d(S,H) = d(S,G) < m$ since $H[S] = G[S]$. Thus, $c(S,H) - d(S,H) = m + 1 - d(S,G) > 1$. Therefore, H is not ultraconnected.

($\Longleftarrow$). Let $S \subseteq V(H)$. There are two cases to consider.

$v \in S$: Note that $d(S \setminus \{v\}, G) \leq d(S,H)$, and $c(S,H) = c(S \setminus \{v\}, G)$. Thus, $c(S,H) - d(S,H) \leq c(S \setminus \{v\}, G) - d(S \setminus \{v\}, G) \leq 1$.

$v \notin S$: Let m be the number of components of $G - S$ whose vertices are all elements of T. Since the component of $H - S$ containing v contains all other components of $G - S$ having a vertex not in T, $c(S,H) = m+1$. By the hypothesis, $m \leq d(S,G)$. Since $d(S,G) = d(S,H)$, we have

$$c(S,H) - d(S,H) = m + 1 - d(S,G) \leq 1.$$

Since each of these cases gives $c(S,H) - d(S,H) \leq 1$, the graph H is ultraconnected. $\square$

Corollary 3 and Theorem 9 are both special cases of Theorem 10.

Notice that constructing an ultraconnected graph with n vertices using Theorem 10 requires an ultraconnected induced subgraph having $n-1$ vertices. We can construct more 6 vertex ultraconnected graphs from 5 vertex ultraconnected graphs using the extra generality of Theorem 10. In particular, we can construct graphs 30, 31, 32, and 50 from $\hat{W}_4$ and graphs 69 and 72 from C_5 [see 2, p. 124].

Summarizing the ultraconnected graphs we can construct on 6 vertices from 5 vertex ultraconnected graphs with the results in this section, we see that we can construct graphs 1, 4, 8, and 28 with Corollary 3 ($G \vee K_1$); graphs 8, 9, 14, 17, 18, and 47 with Theorem 9; and graphs 30, 31, 32, 50, 69, and 72 with Theorem 10. This leaves only graphs 51, 52, and 74, all of which do not have a 5 vertex ultraconnected induced subgraph.

5 Ultraconnectivity and Cartesian Products

In Section 3, the extra connectivity introduced by joining two graphs seemed to be the dominant factor in determining the ultraconnectivity of the resulting graph. In this section, we explore how ultraconnectivity is affected by the Cartesian product. In contrast to before, it seems that the crucial factor here in preserving ultraconnectivity is the number of edges missing in the Cartesian product.

Theorem 11. *Let G be a graph. Let $H = G \times K_n$, $n \geq 2$. If G is ultraconnected, then H is ultraconnected.*

Proof. Let S be a subset of $V(H)$. Let $S_i = \{x \mid (x, i) \in S\}$. Without loss of generality, we can label the vertices of K_n so that $|S_1| \leq |S_2| \leq \cdots \leq |S_n|$. Let $G_i = G[\{x \mid (x, i) \in H\}]$ be the copies of G in H.

Suppose that $|S_1| = 0$. Given two vertices (u, i) and (v, j) in $H - S$, there is an edge in $H - S$ between (u, i) and $(u, 1)$, a path between $(u, 1)$ and $(v, 1)$, and another edge between $(v, 1)$ and (v, j). Thus, there is a path between (u, i) and (v, j) in $H - S$, so $H - S$ is connected. Therefore, $c(S, H) \leq 1$, and so $c(S, H) - d(S, H) \leq 1$.

Now suppose that $|S_1| = 1$. Let $S_1 = \{x\}$. Then since G is ultraconnected, $G_1 - S_1$ is connected. Let (u, i) and (v, j) be vertices in $H - S$ such that $u \neq x$ and $v \neq x$. Then, just as before, there is an edge between (u, i) and $(u, 1)$, and another edge between (v, j) and $(v, 1)$. Since $G_1 - S_1$ is connected, there is a path between $(u, 1)$ and $(v, 1)$ in $G_1 - S_1$. Thus, there is a path between (u, i) and (v, j) in $H - S$. Thus, the vertices $\{(u, i) \mid u \neq x, (u, i) \in V(H - S)\}$ are connected in $H - S$. Let A be the component in $H - S$ containing these vertices. Now, if there were another component in $H - S$, it would need to have vertices only of the form (x, k), with $k \neq 1$. Since we are taking the Cartesian product with K_n, all the vertices $\{(x, k) \mid k \neq 1, (x, k) \in V(H - S)\}$ are connected. Therefore, there are at most two components in $H - S$, and $c(S, H) \leq 2$. If $c(S, H) = 2$, then there is a component of $H - S$ consisting entirely of vertices of the form (x, k), $k \neq 1$. Let (x, m) be one of these vertices in $H - S$. Since G_m is connected, x is adjacent to some vertex in G_m. Let w be a vertex adjacent to x in G_m. Then $(w, m) \in S$, for if not, the component containing (x, m) in $H - S$ would contain (w, m), and thus would be a part of the component A, and therefore $c(S, H) = 1$. Since $(x, 1) \in S$, and $(w, m) \in S$, and $x \neq w$, there is at least one edge missing in $H[S]$. Therefore, $d(S, H) \geq 1$, and $c(S, H) - d(S, H) \leq 2 - 1 = 1$.

Now, suppose that $|S_1| \geq 2$. Since there are n copies of G, we have

$$c(S,H) \leq \sum_{i=1}^{n} c(S_i, G).$$

The edges missing in $H[S]$ are just the edges missing in each copy of G plus the edges missing between copies of G. Thus, $d(S,H) = A + \sum_{i=1}^{n} d(S_i, G)$, where A is the number of edges missing between copies of G in $H[S]$. For any given pair of sets S_i and S_j, $i \neq j$, the number of edges between them in the Cartesian product is at most $\min\{|S_i|, |S_j|\}$. Thus, the number of edges between copies of G in $H[S]$ is at most

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \min\{|S_i|, |S_j|\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |S_i| = \sum_{i=1}^{n-1} |S_i|(n-i)$$

The number of possible edges between two sets S_i and S_j, $i \neq j$, is $|S_i||S_j|$. Thus,

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |S_i||S_j| - \sum_{i=1}^{n-1} |S_i|(n-i) \leq A \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |S_i||S_j|.$$

Now, since $|S_i| \geq 2$ for all i, we have

$$\begin{aligned}
\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |S_i||S_j| - \sum_{i=1}^{n-1} |S_i|(n-i) &= \sum_{i=1}^{n-1} \left(|S_i| \sum_{j=i+1}^{n} |S_j| \right) - \sum_{i=1}^{n-1} |S_i|(n-i) \\
&\geq \sum_{i=1}^{n-1} 2|S_i|(n-i) - \sum_{i=1}^{n-1} |S_i|(n-i) \\
&= \sum_{i=1}^{n-1} |S_i|(n-i) \\
&\geq 2\sum_{i=1}^{n-1} (n-i) = n^2 - n.
\end{aligned}$$

Therefore, $n^2 - n \leq A$.

We can then calculate an upper bound for $c(S,H) - d(S,H)$:

$$\begin{aligned} c(S,H) - d(S,H) &\leq \sum_{i=1}^{n} c(S_i, G) - \sum_{i=1}^{n} d(S_i, G) - A \\ &= \sum_{i=1}^{n} (c(S_i, G) - d(S_i, G)) - A \\ &\leq n - A \leq n - n^2 + n \\ &\leq 0. \end{aligned}$$

Therefore, H is ultraconnected. □

Since the hypercube Q_n can be expressed as successive Cartesian products with K_2, starting with K_1, we have the following corollary.

Corollary 12. *The hypercube Q_n is ultraconnected.*

We note several facts from the proof of Theorem 11 that hold when K_n is replaced by an arbitrary graph J. First, $c(S, G \times J) \leq \sum_{j \in V(J)} c(S_j, G)$ for any subset $S \subseteq V(J)$. Also, in the case $|S_i| \geq 2$ for all $i \in V(J)$, we had $c(S, G \times J) - d(S, G \times J) \leq 0$. This result came when we assumed that $|J| \geq 2$. If $|J| = 1$, then $G \times J = G$, and $c(S, G \times J) - d(S, G \times J) = c(S_1, G) - d(S_1, G) \leq 1$. We state these results as a lemma.

Lemma 13. *Let J be a graph. Let G be an ultraconnected graph with more than one vertex. Let $S \subseteq V(G \times J)$. Then $c(S, G \times J) \leq \sum_{j \in V(J)} c(S_j, G)$.*

In addition, if $|S_i| \geq 2$, $i \in V(J)$ (with S_i defined as in the proof of Theorem 11), then the following are true:

(i) $d(S, G \times J) \geq \sum_{j \in V(J)} d(S_j, G) + |J|^2 - |J|$,

(ii) $c(S, G \times J) - d(S, G \times J) \leq 1$.

Now, we generalize even further, and prove that the Cartesian product preserves ultraconnectivity. Note that since K_1 is ultraconnected, $G \times K_1 = G$, and $K_1 \times J = J$, both G and J must be ultraconnected to preserve ultraconnectivity in general.

Theorem 14. *If G and J are ultraconnected graphs, then $G \times J$ is an ultraconnected graph.*

Proof. Let $H = G \times J$. Let $a = |V(G)|$ and $b = |V(J)|$. Let S be a subset of $V(H)$. Let $S_i = \{x \mid (x, i) \in S\}$. Let $G_i = G[\{x \mid (x, i) \in H\}]$ be the copies of G in H. Label the vertices of J with the numbers $\{1, 2, \ldots, b\}$ such that $|S_1| \leq |S_2| \leq \cdots \leq |S_b|$.

Define the following subsets of the vertices of J:

$$\begin{aligned} I &= \{i \mid |S_i| = 0\}, \\ K &= \{k \mid |S_k| \geq 1\}, \\ M &= \{m \mid |S_m| = 1\}, \\ N &= \{n \mid |S_n| \geq 2\}. \end{aligned}$$

Note that $K = M \dot\cup N$, and $I \dot\cup K = \{1, 2, \ldots, b\}$.

If G or J is complete, the theorem follows from the previous result, so assume that G and J are not complete. The only non-complete ultraconnected graph on 4 or fewer vertices is C_4, therefore $a \geq 4$ and $b \geq 4$.

We prove the result in cases, depending on S.

$N = \emptyset$: In this case, we are choosing at most one vertex out of each copy of G_i, $i = 1, \ldots, b$. Since G is ultraconnected, each copy of G minus a vertex is still connected.

Let (u, v) and (u', v') be vertices in $H - S$. Since J is connected, there is some path $v = v_1 v_2 \cdots v_n = v'$ between v and v' in J. Since only one vertex is missing out of $G_{v_1} - S_{v_1}$, and only one vertex missing out of $G_{v_2} - S_{v_2}$, but $|V(G)| > 2$, there is a vertex in G_{v_1} which is adjacent to a vertex in G_{v_2}. Likewise, there is at least one edge between G_{v_i} and $G_{v_{i+1}}$ for $i = 1, \ldots, n-1$. Now, since $G_{v_i} - S_{v_i}$ is connected, there is a path in G_{v_i}, $i = 2, \ldots, n-1$, between the vertex adjacent to $G_{v_{i-1}}$ and the vertex adjacent to $G_{v_{i+1}}$. Thus, there is a path in $H - S$ between (u, v) and (u', v') by first traveling through G_{v_1} to a vertex adjacent to G_{v_2}, then to G_{v_2}, then through G_{v_2} to a vertex adjacent to G_{v_3}, etc. Thus, $H - S$ is connected, and so $c(S, H) - d(S, H) \leq 1$.

For the rest of the proof, we assume that $N \neq \emptyset$.

$I = \emptyset$, $M = \emptyset$: Then $K = N = \{1, \ldots, b\}$. We are choosing two or more vertices out of every copy of G, i.e., $|S_i| \geq 2$. Therefore, by Lemma 13, $c(S, H) - d(S, H) \leq 1$.

$I = \emptyset$, $M \neq \emptyset$: Since each $G_m - S_m$, $m \in M$, is connected, we have $c(S, H) \leq c(S_N, H_N) + |M|$, where $H_N = G \times (J[N])$ and S_N is S restricted to H_N. For each S_m, $m \in M$, there is at least one missing edge between a vertex in S_m and a vertex in each S_n, $n \in N$. Therefore

$$d(S, H) \geq d(S_N, H_N) + |M| \sum_{n \in N} (|S_n| - 1) \geq d(S_N, H_N) + |M||N|$$

Thus,

$$\begin{aligned} c(S, H) - d(S, H) &\leq c(S_N, H_N) + |M| - (d(S_N, H_N) + |M||N|) \\ &= c(S_N, H_N) - d(S_N, H_N) - (|N| - 1)|M| \end{aligned}$$

Since $H_N = G \times (J[N])$ and S_N satisfy the conditions of Lemma 13, we have $c(S_N, H_N) - d(S_N, H_N) \leq 1$. Thus, $c(S, H) - d(S, H) \leq 1$.

Notice that nowhere in this case did we use that J is ultraconnected, or even connected. The only important connectivity condition we used was that G is ultraconnected.

$I \neq \emptyset$, $K \neq \emptyset$: As above, we define the notation $H_P = G \times J[P]$, $P \subseteq V(J)$.

Suppose $|K| = 1$, and $K = \{k\}$. Then $J - K$ is connected since J is ultraconnected, and therefore $G \times (J - K)$ is connected. Every vertex of $G_k - S_k$ is connected to some vertex in $G \times (J - K)$, and so $H - S$ is connected. Therefore, $c(S, H) = 1$, and $c(S, H) - d(S, H) \leq 1$.

Now suppose that $|K| \geq 2$.

The primary difficulty in this proof is controlling the number of components of $H - S$ which have no vertices in one of G_k, $k \in K$. In the argument below, we use the ultraconnectivity of J to control this number, and get suitable bounds to show that $c(S, H) - d(S, H) \leq 1$.

The components of $H - S$ can be divided into two sets. Let c_k be the number of components of $H - S$ which have a vertex in one of G_k, $k \in K$, and let c_i be the number of other components (i.e., those with vertices only in G_i, $i \in I$). These two sets are disjoint, so $c(S, H) = c_k + c_i$.

Now, $c_k \leq c(S, H_K)$, since there are at most $c(S, H_K)$ components having a vertex in some G_k, $k \in K$. Also, $c(S, H_K) \leq \sum_{k \in K} c(S_k, G)$ by Lemma 13. Therefore,

$$c_k \leq c(S, H_K) \leq \sum_{k \in K} c(S_k, G).$$

If $c_i > 0$, then there are connected components of $H - S$ which contain only vertices of G_i, $i \in I$. Each of these must be separated from the components in $H - S$ containing a vertex in G_k, $k \in K$, by a copy of G that is entirely in S. For if not, since every vertex of the G_i is not in S, and at least one vertex of G_k is not in S, the G_i component would contain a vertex of G_k. Let $L = \{l \in V(J) \mid S_l = V(G)\}$ index the copies of G whose vertices are contained entirely in S. Thus, if $c_i > 0$, then $|L| > 0$.

Since J is ultraconnected, $c(L, J) - d(L, J) \leq 1$. The vertices of each component of $J - L$ are either all in I or not. The components of $J - L$ whose vertices are entirely in I correspond exactly to the components of $H - S$ whose vertices are all in G_i, $i \in I$. Every component in $H - S$ consisting of only vertices in G_i, $i \in I$, corresponds to a component of $J - L$ in which all the vertices are in I. Thus,

$$c_i \leq c(L, J) \leq d(L, J) + 1.$$

We next bound $d(S, H)$. The edges missing in $H[S]$ include the edges missing inside copies of G_k, $k \in K$, plus edges missing between copies of G_l, $l \in L$, plus edges missing between copies of G_l, $l \in L$, and copies of G_t, $t \in K \setminus L$.

The number of edges missing in $H[S]$ inside copies of G_k, $k \in K$ is exactly $\sum_{k \in K} d(S_k, G)$.

Let $l_1, l_2 \in L$. We will count only the missing edges corresponding to missing edges in J. If $l_1 l_2$ is not an edge in J, then $|S_{l_1}||S_{l_2}|$ edges are missing in $H[S]$. Since $V(G_{l_1})$ and $V(G_{l_2})$ are entirely in S, and there are a vertices in each copy of G, there are a^2 edges missing between G_{l_1} and G_{l_2} in $H[S]$.

Let $l \in L$ and $t \in K \setminus L$. Since there are a vertices in S from G_l, there are at least $a|S_t| - |S_t| = (a-1)|S_t|$ edges missing between G_l and G_t. This is a minimum when $|S_t| = 1$. Thus, there are at least $(a-1)$ edges missing between G_l and G_t in $H[S]$.

Thus, we have

$$d(S, H) \geq \sum_{k \in K} d(S_k, G) + a^2 d(L, J) + |L|(|K| - |L|)(a-1). \tag{4}$$

If $|K| - |L| = 0$, then $K = L$. Then every component of $H - S$ corresponds exactly to a component of $J - L$. Thus, $c(S, H) = c(L, J)$. Also, since a missing edge in $J[L]$ corresponds to at least a^2 missing edges in $H[S]$, we have $d(S, H) \geq d(L, J)$. Therefore, since J is ultraconnected,

$$c(S, H) - d(S, H) \leq c(L, J) - d(L, J) \leq 1.$$

If $|L| = 0$, then $c_i = 0$ (see the paragraph defining L on page 16). Thus, $c(S, H) = c_k \leq c(S, H_K)$. Also, since $S \subseteq H_K$, we have $d(S, H) = d(S, H_K)$. Therefore, $c(S, H) - d(S, H) \leq c(S, H_K) - d(S, H_K)$. This then reduces to one of the cases $I = \emptyset$, $M = \emptyset$ or $I = \emptyset$, $M \neq \emptyset$ above. In either case,

$$c(S, H) - d(S, H) \leq c(S, H_K) - d(S, H_K) \leq 1.$$

Assume then that $0 < |L| < |K|$.

Either $|L| = |K| - 1$ or $|L| = 1$ minimizes the last term of equation (4). Also, since $a \geq 4$, we have $a - 1 \geq 3$. Therefore,

$$\begin{aligned} d(S, H) &\geq \sum_{k \in K} d(S_k, G) + a^2 d(L, J) + (|K| - 1)(a-1) \\ &\geq \sum_{k \in K} d(S_k, G) + a^2 d(L, J) + 3(|K| - 1) \end{aligned}$$

Putting our results together, and using the fact that G is ultraconnected (so

$c(S_k, G) - d(S_k, G) \leq 1$, $k \in K$), we have

$$\begin{aligned}
c(S,H) - d(S,H) &= c_k + c_i - d(S,H) \\
&\leq \sum_{k \in K} c(S_k, G) + d(L,J) + 1 \\
&\qquad - \left(\sum_{k \in K} d(S_k, G) + a^2 d(L,J) + 3(|K| - 1) \right) \\
&= \sum_{k \in K} (c(S_k, G) - d(S_k, G)) + (1 - a^2) d(L,J) - 3|K| + 4 \\
&\leq |K| + (1 - a^2) d(L,J) - 3|K| + 4 \\
&\leq 4 - 2|K| + (1 - a^2) d(L,J)
\end{aligned}$$

Since $|K| \geq 2$, we have $4 - 2|K| \leq 0$. Since $a \geq 4$, we have $(1 - a^2)d(L,J) \leq 0$. Therefore,

$$c(S,H) - d(S,H) \leq 4 - 2|K| + (1 - a^2)d(L,J) \leq 0$$

We have covered all possible cases. Therefore, H is ultraconnected. □

6 Ultraconnectivity and Regular Graphs

To introduce this section, we determine when an $(n-3)$-regular graph is ultraconnected. Let G be an $(n-3)$-regular graph on n vertices. Let $S \subseteq V(G)$ such that $c(S,G) > 1$. Such an S exists, for since G is not the complete graph, there are two vertices u and v which are not adjacent. Thus, $V(G) \setminus \{u, v\}$ is an example of an S which will give us $c(S,G) > 1$.

Now, $G - S = A \cup B$ is the union of two graphs. Therefore, $(G - S)^c = A^c \vee B^c$ is the join of two graphs. Note that since G^c is an $n - 1 - (n - 3) = 2$-regular graph, and $(G - S)^c$ is a subgraph of G^c, we have $\Delta((G - S)^c) \leq \Delta(G^c) = 2$. Since $(G - S)^c$ is the join of two graphs, the number of vertices in each of A and B cannot exceed 2. Thus, A and B could each be one of K_1, K_2, or K_2^c, the only graphs on two or fewer vertices.

Let p be the number of vertices in $G - S$. Let c be the number of components in $G - S$, and e be the number of edges in $G - S$. Then $|S| = n - p$. Using Lemma 15 on the facing page, we can calculate $d(S,G)$ from each of the possible S by letting $k = n - 3$, and noting that the number of missing edges in $G[S]$ is the difference between the possible number of edges and the number of edges given by Lemma 15. Doing this gives us $d(S,G) = \binom{|S|}{2} - (n-3)(n-p) + e - n(n-3)/2$.

Thus, if $K_1 \cup K_1$ is the only possible S, then $\gamma(G) = 5 - n$, and G is ultraconnected iff $n \geq 4$. However, if any other possibility for S exists in G, then $\gamma(G) = 6 - n$ and G is ultraconnected iff $n \geq 5$.

$G-S$	p	c	e	$d(S,G)$	$c(S,G)-d(S,G)$
$K_1 \cup K_1$	2	2	0	$n-3$	$5-n$
$K_1 \cup K_2$	3	2	1	$n-4$	$6-n$
$K_1 \cup K_2^c$	3	3	0	$n-3$	$6-n$
$K_2 \cup K_2$	4	2	2	$n-4$	$6-n$
$K_2 \cup K_2^c$	4	3	1	$n-3$	$6-n$
$K_2^c \cup K_2^c$	4	4	0	$n-2$	$6-n$

Table 1: $(n-3)$-regular configurations

A simple check of all $(n-3)$-regular graphs for $4 \leq n < 5$ (namely $C_4^c = K_2 \cup K_2$), shows that G is ultraconnected iff $n \geq 5$.

In order to calculate $c(S,G) - d(S,G)$ in the example above, we needed to know how many edges were in $G[S]$ for a particular subset $S \subseteq V(G)$. We use the regularity of G to obtain this value in terms of the number of edges in $G-S$.

Lemma 15. *Let G be a k-regular graph and let $S \subseteq V(G)$, with cardinality $|S| = s$. Let e be the number of edges in $G-S$. Then $G[S]$ has $ks + e - \frac{1}{2}kn$ edges.*

Proof. Each vertex of $G-S$ is adjacent to k edges. Since there are e edges within $G-S$, this accounts for $2e$ of the needed adjacent edges. Thus, the number of edges between $G-S$ and $G[S]$ is $k(n-s)-2e$. Likewise, each of the s vertices in $G[S]$ is adjacent to k edges. Since $k(n-s)-2e = kn-ks-2e$ of these edges are from $G-S$, there must be $ks-(kn-ks-2e)$ adjacencies in $G[S]$ to edges in $G[S]$. Since each edge is counted twice, the number of edges in $G[S]$ is

$$\frac{1}{2}(ks-(kn-ks-2e)) = ks + e - \frac{1}{2}kn.$$

□

The technique used in the example above can be generalized to determine lower bounds for the number of vertices in an $(n-r)$-regular graph sufficient to guarantee ultraconnectivity.

Theorem 16. *Let G be an $(n-r)$-regular graph. Then depending on the value of r:*

$r = 1$: *G is ultraconnected.*

$r = 2$: *If $n \geq 5$, then G is ultraconnected.*

$r \geq 3$: *If $n \geq 3r$, then G is ultraconnected.*

Proof. If $r = 1$, then $G = K_n$, and so is ultraconnected by Corollary 3. Assume then that $r > 1$.

Let $S \subseteq V(G)$ such that $c(S,G) > 1$ (since there are nonadjacent vertices in G because $r > 1$, this is possible). Then $G - S = A \cup B$ is the union of two graphs. Thus, $(G-S)^c = A^c \vee B^c$ is the join of two graphs. Note also that $(G-S)^c$ is an induced subgraph of G^c, an $n-1-(n-r)=(r-1)$-regular graph. Thus, the number of vertices in A or B cannot exceed $r-1$. Thus, the number of vertices in $(G-S)^c$, and hence in $G-S$, cannot exceed $2(r-1)$. Let p be the number of vertices in $G-S$. Then $|S|=n-p$.

Let c be the number of components in $G-S$, and let e be the number of edges in $G-S$. Note that $2 \le c \le 2(r-1)$ and $0 \le e \le 2\binom{r-1}{2} = (r-2)(r-1)$. From Lemma 15, the number of edges in $G[S]$ is $(n-r)(n-p)+e-\frac{1}{2}(n-r)n$, and thus

$$\begin{aligned}
d(S,G) &= \binom{n-p}{2} - (n-r)(n-p) - e + \frac{1}{2}(n-r)n \\
&= \frac{1}{2}(n-p)(n-p-1) - n^2 + rn + pn - rp - e + \frac{1}{2}n^2 - \frac{1}{2}rn \\
&= \frac{1}{2}(n^2 - np - n - pn + p^2 + p - n^2 + rn + 2pn - 2rp - 2e) \\
&= \frac{1}{2}(-n + p^2 + p + rn - 2rp - 2e) \\
&= -e - \frac{1}{2}p(2r-p-1) + \frac{1}{2}n(r-1).
\end{aligned}$$

Therefore

$$c(S,G) - d(S,G) = c + e + \frac{1}{2}p(2r-p-1) - \frac{1}{2}n(r-1). \tag{5}$$

Maximizing $c(S,G)-d(S,G)$ by choosing different sets S corresponds to maximizing by varying c, e, and p. Therefore we need to maximize the first three terms of (5). From above, we have $c \le 2(r-1)$ and $e \le (r-2)(r-1)$. The maximum of $\frac{1}{2}p(2r-p-1)$ occurs when the derivative with respect to p is zero, which is when $p = r - \frac{1}{2}$. Thus,

$$\begin{aligned}
c(S,G) - d(S,G) &= c + e + \frac{1}{2}p(2r-p-1) - \frac{1}{2}n(r-1) \\
&\le 2(r-1) + (r-2)(r-1) + \frac{1}{2}\left(r^2 - r + \frac{1}{4}\right) - \frac{1}{2}n(r-1) \\
&= \frac{1}{2}(r-1)(3r-n) + \frac{1}{8}.
\end{aligned}$$

Setting this last expression less than or equal to one and solving for n then gives us the desired bound to make $c(S,G)-d(S,G) \le 1$:

$$\begin{aligned}
\frac{1}{2}(r-1)(3r-n) + \frac{1}{8} &\le 1 \\
3r - n &\le \frac{7}{4(r-1)} \\
n &\ge 3r - \frac{7}{4(r-1)}.
\end{aligned}$$

For $r=2$, the bound is $n \ge 5$. For $r \ge 3$, the fractional part of the bound is less than one. Since n is an integer, we can then ignore the fractional part. Thus, for $r \geq 3$, $n \geq 3r$ is a sufficient bound to guarantee ultraconnectivity. □

As examples of Theorem 16, all $n-3$ regular graphs (e.g., C_n^c) with $n \geq 9$ are ultraconnected. In addition, all $n-4$ regular graphs with $n \geq 12$ are ultraconnected.

We can make the bound strict by considering the exact combinations of p, e, and c that arise, as we did in our first example. Doing this, for example, leads to all $(n-2)$-regular graphs being ultraconnected for $n \geq 4$. Various strict lower bounds obtained in this way are compared to bounds given by Theorem 16 in Table 2.

r	Strict Bound	Thm 16 Bound
1	$n \geq 0$	$n \geq 0$
2	$n \geq 4$	$n \geq 5$
3	$n \geq 5$	$n \geq 9$
4	$n \geq 20/3$	$n \geq 12$
5	$n \geq 17/2$	$n \geq 15$
6	$n \geq 52/5$	$n \geq 18$
7	$n \geq 37/3$	$n \geq 21$
8	$n \geq 15$	$n \geq 24$

Table 2: Lower bounds sufficient for ultraconnectivity of $(n-r)$-regular graphs

7 Enumeration of Ultraconnected Graphs

Using computer code we wrote, listed in Appendix A, in conjuction with Brenden McKay's nauty program [18], we exhaustively constructed and enumerated all ultraconnected graphs up to 11 vertices. The results of the enumeration are summarized in Table 3, along with a comparison to the numbers of connected graphs. We have also published the sequence in Sloane's Online integer sequence database [21, sequence A046082]. The data files of all constructed ultraconnected graphs are available from the author on request.

We note several interesting things. The ratio of ultraconnected to connected graphs steadily declines at first for small n, but then increases sharply. Asymptotically, almost all graphs are connected [see 9, p. 206]. This empirical evidence suggests that asymptotically, all connected graphs are ultraconnected.

We also found several interesting properties in our construction. Every vertex-transitive graph up to 11 vertices is ultraconnected. Also, up to 11 vertices, every connected regular graph without a clique separator is ultraconnected, although there is a 23 vertex regular graph not having a clique separator that is not ultraconnected.

n	# of ultraconnected graphs	# of connected graphs	% ultraconnected graphs
1	1	1	100 %
2	1	1	100 %
3	1	2	50 %
4	2	6	33.33 %
5	4	21	19.05 %
6	19	112	16.96 %
7	139	853	16.30 %
8	2 319	11 117	20.86 %
9	77 423	261 080	29.65 %
10	4 909 331	11 716 571	41.90 %
11	554 491 273	1 006 700 565	55.08 %

Table 3: Number of Ultraconnected Graphs on n vertices

8 Criticality and the Lovász Parameter

8.1 The Lovász Theta Parameter

Using the Lovász theta function, we can deduce the criticality of certain forms of matrices. In this section, we derive results about the criticality of graphs using results about the theta function from Lovász [17] and Knuth [14].

In 1979, Lovász introduced the theta parameter of a graph, $\vartheta(G)$ [17]. This parameter has been extensively studied since then [e.g., see 7, 8, 14, 16].

In Knuth [14], we find that the Lovász parameter of a graph can be represented as follows.

Definition 27. We say an n-by-n matrix A is a feasible matrix for an n-vertex graph G and a positive real vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ if

(i) A is real and symmetric;

(ii) $A_{vv} = w_v$ for $v = 1, \ldots, n$;

(iii) $A_{uv} = \sqrt{w_u w_v}$ whenever $uv \notin E(G)$

Define the parameter $\vartheta(G, w)$ as

$$\vartheta(G, w) = \min\{\lambda_1(A) \mid A \text{ is a feasible matrix for } G \text{ and } w\}.$$

A proof that this minimum exists for the case $w = (1, 1, \ldots, 1)$ is given in Riddle [20, p. 10].

There are several alternate definitions for $\vartheta(G)$. The above definition involving feasible matrices is called $\vartheta_2(G)$ in Knuth [14]. If $w = (1, 1, \ldots, 1)$, then we abbreviate the function $\vartheta(G, w)$ as $\vartheta(G)$.

We summarize a few of the known results [see 14] for the parameter below.

Minimum: The minimum ϑ for all graphs on n vertices with a given w is $\vartheta_{\min} = \vartheta(K_n, w) = \max\{w_1, \ldots, w_n\}$. In particular, $\vartheta(K_n) = 1$.

Maximum: The maximum ϑ for all graphs on n vertices with a given w is $\vartheta_{\max} = \vartheta(K_n^c, w) = w_1 + \cdots + w_n$. In particular, $\vartheta(K_n^c) = n$.

Union: Let $G = G_1 \cup G_2$. Then $\vartheta(G) = \vartheta(G_1) + \vartheta(G_2)$.

Join: Let $G = G_1 \vee G_2$. Then $\vartheta(G) = \max\{\vartheta(G_1), \vartheta(G_2)\}$.

8.2 Criticality of C_n^c

We first show that C_5^c is critical, followed by the generalization to C_n^c.

To establish the criticality of C_5^c, we need to exhibit a 5-by-5 C_5^c-partial matrix A such that every 4-by-4 principal submatrix of A has a positive definite completion, but A does not.

Theorem 17. *The graph C_5^c is critical.*

Proof. Let A be the C_5^c-partial matrix

$$A = \begin{bmatrix} d & x_{12} & -1 & -1 & x_{15} \\ x_{12} & d & x_{23} & -1 & -1 \\ -1 & x_{23} & d & x_{34} & -1 \\ -1 & -1 & x_{34} & d & x_{45} \\ x_{15} & -1 & -1 & x_{45} & d \end{bmatrix},$$

where each x_{ij} represents an unspecified entry. Let B be the C_5^c-partial matrix

$$B = (d+1)I - A = \begin{bmatrix} 1 & -x_{12} & 1 & 1 & -x_{15} \\ -x_{12} & 1 & -x_{23} & 1 & 1 \\ 1 & -x_{23} & 1 & -x_{34} & 1 \\ 1 & 1 & -x_{34} & 1 & -x_{45} \\ -x_{15} & 1 & 1 & -x_{45} & 1 \end{bmatrix}.$$

Let $\tilde{A}$ be a completion of A, and $\tilde{B}$ be the corresponding completion of B. Then the eigenvalues of $\tilde{A}$ are $d + 1 - \lambda_i(\tilde{B})$ for $i = 1, \ldots, 5$. Since $\lambda_1(\tilde{B})$ is the largest eigenvalue of $\tilde{B}$, $\lambda_n(\tilde{A}) = d+1-\lambda_1(\tilde{B})$. Therefore, A does not have a positive definite completion if and only if $d+1-\lambda_1(\tilde{B}) \le 0$ for all possible completions $\tilde{B}$ of B. Note that since B is a feasible matrix for the graph C_5, $\min_{\tilde{B}} \lambda_1(\tilde{B}) = \vartheta(C_5)$. Therefore, A does not have a positive definite completion if and only if $d + 1 - \vartheta(C_5) \le 0$. It is well known that $\vartheta(C_5) = \sqrt{5}$ [see 17, p. 2] (for alternate proofs, see [14, p. 26] and [20, Section 4]). Therefore, A does not have a positive definite completion if and only if $d + 1 - \sqrt{5} \le 0$.

Since C_5^c is vertex transitive, all principal submatrices of A are of the same form up to permutation. Thus, we only need to consider one of A_1, A_2, etc. We consider the completion of the matrix A_5,

$$A_5 = (d+1)I - B_5 = \begin{bmatrix} d & x_{12} & -1 & -1 \\ x_{12} & d & x_{23} & -1 \\ -1 & x_{23} & d & x_{34} \\ -1 & -1 & x_{34} & d \end{bmatrix}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be the sets of completions for A_5 and B_5 respectively. Since B_5 is a feasible matrix for the perfect graph P_4, we know by the Sandwich Theorem [see 14] for the Lovász theta function that

$$\min_{\tilde{B}_5 \in \mathcal{B}} \lambda_1(\tilde{B}_5) = \vartheta(P_4) = \alpha(P_4) = 2$$

[see 20, Corollary 1]. Since any completion $\tilde{A}_5 \in \mathcal{A}$ satisfies the relation $\lambda_n(\tilde{A}_5) = d+1-\lambda_1(\tilde{B}_5)$ for a corresponding $\tilde{B}_5 \in \mathcal{B}$, A_5 has a positive definite completion if and only if $d+1-2>0$. A possible completion would then be the completion $\tilde{A}_5$ corresponding to the $\tilde{B}_5$ satisfying $\lambda_1(\tilde{B}_5) = \vartheta(B) = 2$.

Combining these two conditions, the entries of A are critical data for C_5^c if and only if $1 < d \leq \sqrt{5}-1$. □

Theorem 17 can be extended to a more general case.

Theorem 18. *The graph C_n^c is critical for odd $n \geq 5$.*

Proof. Let A be the C_n^c-partial matrix such that all the specified entries off of the diagonal are -1 and the diagonal entries are d. Let B be the C_n^c-partial matrix $B = (d+1)I - A$.

As above, since B is a feasible matrix for C_n, A has no positive definite completion if and only if $d+1-\vartheta(C_n) \leq 0$. From Knuth [14, Section 22], we know

$$\vartheta(C_n) = \begin{cases} \frac{n\cos(\pi/n)}{1+\cos(\pi/n)} & \text{if } n \text{ odd,} \\ n/2 & \text{if } n \text{ even} \end{cases}$$

(Lovász proves the odd case in [17, p. 5], and Riddle gives an alternate proof in [20, Section 4]). Thus, if n is odd, A has no positive definite completion if and only if $d \leq \frac{n\cos(\pi/n)}{1+\cos(\pi/n)} - 1$.

As in Theorem 17, since C_n is vertex transitive, all principal submatrices of A have the same form up to permutation. Thus, we only need to consider one principal submatrix of A, say $A_1 = (d+1)I - B_1$. Since B_1 is a feasible matrix for the perfect graph P_{n-1}, $\vartheta(B_1) = \alpha(P_{n-1}) = \frac{n-1}{2}$. Thus, if $\tilde{A}_1$ and $\tilde{B}_1$ are corresponding completions of A_1 and B_1, then

$$\lambda_n(\tilde{A}_1) = d+1-\lambda_1(\tilde{B}_1) \leq d+1-\vartheta(P_{n-1}) = d+1-\frac{n-1}{2},$$

and the inequality is an equality for a some particular completion $\tilde{B}_1$. Thus, A_1 has a positive definite completion if and only if $d > \frac{n-1}{2} - 1$.

Thus, if n is odd, A is critical data for C_n^c if and only if

$$\frac{n-1}{2} - 1 < d \leq \frac{n\cos(\pi/n)}{1+\cos(\pi/n)} - 1.$$

This can only happen when $\frac{n-1}{2} < \frac{n\cos(\pi/n)}{1+\cos(\pi/n)}$. By Lemma 19 below, this is true if and only if $n > 3$. Thus, A can provide critical data for C_n^c for odd $n \geq 5$.

Thus, C_n^c is critical for odd $n \geq 5$. □

We note that this approach will not always work, because the ϑ parameter will not always decrease when a vertex is deleted from the graph. An example is C_n with n even. In this case, $\vartheta(C_n)$ is $n/2$. By the method above, A provides critical data for C_n^c iff $\vartheta(P_{n-1}) - 1 < d \leq \vartheta(C_n) - 1$. However, because $n-1$ is odd, $\vartheta(P_{n-1}) = \lceil \frac{n-1}{2} \rceil$. This is true if and only if $\lceil \frac{n-1}{2} \rceil < \frac{n}{2}$, which is never true.

Lemma 19. *For $n \in \mathbb{N}$, the following is true if and only if $n > 3$:*

$$\frac{n-1}{2} < \frac{n\cos(\pi/n)}{1+\cos(\pi/n)}.$$

Proof. First, we reduce the inequality:

$$\begin{aligned} &\frac{n-1}{2} < \frac{n\cos(\pi/n)}{1+\cos(\pi/n)} \\ \iff & n-1+(n-1)\cos(\pi/n) < 2n\cos(\pi/n) \\ \iff & \frac{n-1}{n+1} < \cos(\pi/n) \\ \iff & 1-\cos(\pi/n) < \frac{2}{n+1} \end{aligned}$$

Now, $1-\cos(\pi/n) = \int_0^{\pi/n} \sin x \, dx$. Since $\sin x$ is concave down on $[0, \pi/n]$ when $n \geq 1$, the integral is less than its midpoint approximation, $\frac{\pi}{n}\sin\left(\frac{\pi}{2n}\right)$. Also, $\sin\left(\frac{\pi}{2n}\right) < \frac{\pi}{2n}$ for $n \geq 1$. Therefore,

$$1-\cos(\pi/n) = \int_0^{\pi/n} \sin x \, dx < \frac{\pi}{n}\sin\left(\frac{\pi}{2n}\right) < \frac{\pi^2}{2n^2}$$

Solving the inequality $\frac{\pi^2}{2n^2} < \frac{2}{n+1}$ using the quadratic formula yields

$$n > \frac{\pi^2 + \pi\sqrt{\pi^2+16}}{8} \approx 3.231.$$

Thus, $1-\cos(\pi/n) < 2/(n+1)$ when $n > 3$. A simple check with $n = 1$, $n = 2$, and $n = 3$ shows that this bound is tight, and the result follows. □

8.3 General Theorems

The techniques used in the proofs of Theorems 17 and 18 generalize.

Theorem 20. *Let G be a graph such that $\vartheta(G^c) > \vartheta(G^c - v)$ for every $v \in V(G)$. Then G is critical.*

Proof. Let A be the G-partial matrix such that all the specified entries off of the diagonal are -1 and the diagonal entries are d. Let B be the G-partial matrix $B = (d+1)I - A$. Then B is a feasible matrix for G^c. As in the previous proofs, A provides critical data for G if and only if $\vartheta(G^c - v) - 1 < d \leq \vartheta(G^c) - 1$ for every vertex $v \in V(G)$. This can happen if and only if $\vartheta(G^c - v) < \vartheta(G^c)$ for every $v \in V(G)$. □

This result allows us to provide alternate proofs for other known facts.

Corollary 21. *The graph K_n is critical.*

Proof. Let $G = K_n^c$. Then $\vartheta(G) = n$, and $\vartheta(G - v) = n - 1$. Thus, by Theorem 20, $G^c = K_n$ is critical. □

Theorem 22. *The graph C_n is critical for odd n.*

Proof. We know that $C_1 = K_1$ is critical. Assume n is odd and $n \geq 3$. From Knuth [14], $\vartheta(C_n^c) = \frac{1+\cos(\pi/n)}{\cos(\pi/n)} = \sec(\pi/n) + 1$. Also, $C_n^c - v = P_{n-1}^c$, $v \in V(C_n^c)$. Since P_{n-1} is a perfect graph, P_{n-1}^c is also perfect, and $\alpha(P_{n-1}^c) = \omega(P_{n-1}) = 2$. Since $n \geq 3$, $\sec(\pi/n) > 1$, so $\vartheta(C_n^c) > 2 = \vartheta(C_n^c - v)$ for every vertex $v \in V(C_n^c)$. Therefore, by Theorem 20, C_n is critical for odd n. □

Barrett, Johnson, and Loewy [2, p. 120] use another method to prove that C_n is critical for all $n \geq 3$. To summarize their proof, C_3 is critical because $C_3 = K_3$. For $n \geq 4$, C_n is not chordal, so by the chordal theorem of Grone et al. [6], there is an associated partial matrix A which does not have a positive definite completion. However, every principal submatrix of A is associated with P_{n-1}, which is chordal. Therefore, again by the chordal theorem of Grone et al., the every principal submatrix of A has a positive definite completion. Therefore, C_n is critical for all $n \geq 3$.

We can also prove that C_n is ultraconnected using only graph-theoretic reasons.

Theorem 23. *The graph C_n ($n \geq 3$) is ultraconnected.*

Proof. Let $S \subseteq V(C_n)$. If $S = \emptyset$, then $c(S, C_n) - d(S, C_n) \leq 1$ because C_n is connected. Assume then that $S \neq \emptyset$ and $S \neq V(C_n)$.

We induct on $c(S, C_n)$. If $c(S, C_n) = 1$, then $c(S, C_n) - d(S, C_n) \leq 1$.

We show that increasing $c(S, C_n)$ by one also increases $d(S, C_n)$ by at least one. Since $S \neq \emptyset$, each component of $C_n - S$ is a path. Thus, the only way to increase $c(S, C_n)$ is to add to S an interior vertex of a path in $C_n - S$. Let S' be the $S \cup \{v\}$, where v is an interior vertex of a path in $C_n - S$. The vertex v will then not be connected to any other vertex in S, and thus will be a distinct component of $C_n[S']$. Since $C_n[S] \neq \emptyset$, $d(S, C_n)$ must also increase by at least one. Thus, if $c(S, C_n)$ increases by one, $d(S, C_n)$ also increases by at least one. By induction, $c(S, C_n) - d(S, C_n) \leq 1$ always. □

Using Theorem 20 and the behavior of ϑ with respect to unions of graphs, listed on page 23, we have the following theorem.

Theorem 24. *Let $G = G_1 \vee G_2$. If $\vartheta(G_1^c - v_1) < \vartheta(G_1^c)$, and $\vartheta(G_2^c - v_2) < \vartheta(G_2^c)$ for every $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, then G is critical.*

Proof. Since $G^c = G_1^c \cup G_2^c$, we have $\vartheta(G^c) = \vartheta(G_1^c) + \vartheta(G_2^c)$. If $v \in V(G)$, then $v \in V(G_1)$ or $v \in V(G_2)$. Thus, $\vartheta(G^c - v) < \vartheta(G^c)$, since one of the summands will decrease. Therefore, by Theorem 20, G is critical. □

Theorem 24 enables us to easily prove other classes of graphs are critical. Let T be the set of C_n (n odd), C_n^c (odd $n > 5$), and K_n. Then the graphs in T are critical, and finite joins of graphs from T are critical.

9 Open Questions

We list here several of the open questions developed in the research.

Question 1. In Section 8, we assumed that $w = (1, 1, \ldots, 1)$. This allowed us to prove that $\vartheta(G^c, w) > \vartheta(G^c - v, w)$ for certain graphs G and all $v \in V(G)$. This in turn gave us critical data for these graphs. What happens when we change w? In particular, can we find critical data for any critical graph G just by varying w?

The answer here may be negative—only varying w seems too restrictive. The feasible matrices for G and w have all specified entries determined by w. However, in the general completion problem, there need not be any such relationship between entries of the matrix.

Question 2. Is every vertex transitive graph critical? Is every vertex transitive graph ultraconnected?

Up to 11 vertices, every vertex-transitive graph is ultraconnected.

Question 3. Asymptotically, how many ultraconnected graphs are there?

For $n \leq 7$, the ratio of ultraconnected graphs to connected graphs decreases steadily. However, for $n > 7$, the proportion rapidly jumps beyond 50%.

Question 4. When is the complement of a graph critical or ultraconnected?

In this thesis, we have shown a set of graphs in which criticality is preserved under taking complements (i.e., C_n for odd $n \geq 5$).

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A Source Code Listing

Included in this appendix are the Mathematica, Magma, and C source code listings used to check the ultraconnectivity of graphs.

A.1 Mathematica Source Code

The Mathematica source uses version 2.0.0 of the package *Combinatorica* by Steven Skiena and Sriram Pemmaraju [19]. The code was used in Mathematica 4.0.1.0.

```
subset[n_] := Subsets[Range[n]]

cc[graph_, vertices_] :=
  DeleteVertices[graph, vertices] // ConnectedComponents // Length

dd[graph_, vertices_] :=
  Binomial[Length[vertices], 2] -
    Length[Edges[InduceSubgraph[graph, vertices]]]

ultraconnectedsubsetQ[graph_, vertices_] :=
  cc[graph, vertices] - dd[graph, vertices] <= 1

ultraconnecteddegree[graph_] :=
  Apply[Max,
    Map[cc[graph, #] - dd[graph, #] &,
          subset[Length[Vertices[graph]]]]]

ultraconnectedQ[graph_] :=
  Apply[And,
    Map[ultraconnectedsubsetQ[graph, #] &,
          subset[Length[Vertices[graph]]]]]
```

A.2 Magma Source Code

The Magma code is written for Magma 2.7-2 [3].

```
cc:=function(graph,vertices)
  RemoveVertices(~graph,Setseq(vertices));
```

```
  return #Components(graph);
end function;

dd:=function(graph,vertices)
  return Binomial(#vertices, 2) - \
     #Edges(sub<graph|ChangeUniverse(vertices,VertexSet(graph))>);
end function;

ucsubsetQ:=function(graph,vertices)
  return cc(graph,vertices) - dd(graph,vertices) le 1;
end function;

ucgraphQ:=function(graph)
  return forall{x : x in {n : n in Subsets({1..#VertexSet(graph)}) \
                            | #n lt #VertexSet(graph)} \
                  | (ucsubsetQ(graph,x))};
end function;
```

A.3 C Source code

These are the pertinent modifications (written by the author) used in Brenden McKay's nauty program [18] to check for ultraconnectivity of a graph. This is by far the quickest way the author has seen for checking ultraconnectivity of graphs.

```c
/* For every subset, check the inequality */
int ultraconnected(graph *g, int m, int n)
{
  unsigned long int i;
  int j;
  unsigned long int max;
  unsigned long int graphmask;

#if MAXN
  setword sub[MAXM];       /* The set */
  setword subcomp[MAXM];   /* The complemented set */
#else
  DYNALLSTAT(set,sub,sub_sz);
  DYNALLSTAT(set,subcomp,subcomp_sz);
```

```
  DYNALLOC1(set,sub,sub_sz,m,"ultraconnected");
  DYNALLOC1(set,subcomp,subcomp_sz,m,"ultraconnected");
#endif

  max=1<<(n+1);
  graphmask= ~(BITMASK(n-1));

  for(i=0;i<max;i++)
    {
      for (j = 0; j < m; ++j)
        {
          /* Since the bits are stored from left to right, the set bits
             are at the high end of the word on little-endian machines */
          sub[j] = (i<<(WORDSIZE-n))>>(WORDSIZE*j) & graphmask;
          subcomp[j]=~sub[j] & graphmask;
        }

      if((components(g,m,n,subcomp)-edgesmissing(g,m,n,sub))>1)
        return FALSE;
    }

  return TRUE;
}

/**********************************************************************/
/* Count the number of edges missing from the subset of a graph to the
   complete graph*/
/**********************************************************************/
int edgesmissing(graph *g, int m, int n, set* sub)
{
  set *gw;
  int i,j;
  int edges;
  int subsize;

#if MAXN
  setword subw[MAXM];
#else
  DYNALLSTAT(set,subw,subw_sz);
  DYNALLOC1(set,subw,subw_sz,m,"issubconnected");
#endif

  edges=0;
```

```
  subsize = 0;
  for (i = 0; i < m; ++i) subsize += (sub[i] ? POPCOUNT(sub[i]) : 0);

  if(subsize <= 1)
    return 0;

  /* Count the number of edges in the subset */
  /* For each element of sub, POPCOUNT the number of edges, then divide by 2. */

  for (i = -1; (i = nextelement(sub,m,i)) >= 0;)
    {
      gw = GRAPHROW(g,i,m);
      for (j = 0; j < m; ++j)
          edges += POPCOUNT(gw[j] & sub[j]);
    }

  /* Return Choose(n,2)-edges = n(n-1)/2 - edges*/
  return (subsize*(subsize-1)-edges)/2;
}

/************************************************************** JNG */
/* Count connected components in the graph minus the set sub-pick the
   first vertex, trace through.  If we get all the vertices, then
   fine.  If not, increment the connected components and pick a new
   vertex not hit already. */
/*********************************************************************/

/* n is number of vertices, m is number of edges. sub contains the
   subset we want to use. */

int components(graph *g, int m, int n, set* sub)
{
  int i, head, tail, w, subsize, visitednodes;
  int components;

  set *gw;
#if MAXN
  int queue[MAXN],visited[MAXN];
  setword subw[MAXM];
#else
  DYNALLSTAT(int,queue,queue_sz);
  DYNALLSTAT(int,visited,visited_sz);
  DYNALLSTAT(set,subw,subw_sz);
```

```
  DYNALLOC1(int,queue,queue_sz,n,"issubconnected");
  DYNALLOC1(int,visited,visited_sz,n,"issubconnected");
  DYNALLOC1(set,subw,subw_sz,m,"issubconnected");
#endif

  visitednodes=0;
  components=0;
  subsize = 0;

  for (i = 0; i < m; ++i)
    subsize += (sub[i] ? POPCOUNT(sub[i]) : 0);

  if (subsize <= 1)
    return 1;

  for (i = 0; i < n; ++i)
    visited[i] = 0;

  /* Seed the queue value to pick the first element */
  queue[0]=-1;

  /* while we still have nodes left */
  while(visitednodes < subsize)
    {
      i = nextelement(sub,m,queue[0]);
      while((i>=0) && (visited[i]))
        {
          i = nextelement(sub,m,i);
        }

      if(i==-1) break;

      queue[0]=i;
      components++;

      visited[i] = 1;
      visitednodes++;

      head = 0;
      tail = 1;
      while (head < tail)
        {
```

```
          w = queue[head++];
          gw = GRAPHROW(g,w,m);
          for (i = 0; i < m; ++i) subw[i] = gw[i] & sub[i];

          for (i = -1; (i = nextelement(subw,m,i)) >= 0;)
            {
              if (!visited[i])
                {
                  visited[i] = 1;
                  queue[tail++] = i;
                  visitednodes++;
                }
            }
        }
    }

  return components;
}
```