1 Flowcharting a Function

A principle in understanding how a complicated system works is to understand how the parts behave and how those parts are connected to form the complicated system. That is how we will analyze how a function can be moved and stretched.

We can think of a function as an assembly line of actions that are performed on inputs and which yield outputs. To illustrate, consider the function

\[ f(x) = 2x + 3. \]

Given an input \( x \), we get the output \( f(x) \) by first multiplying by 2 and then adding 3. For instance, if the input is \( x = 4 \), then first we multiply by 2 (getting 8) and then add 3 to that (getting 11). Thus, \( f(4) = 11 \).

You can think of this assembly line as the things you would do to evaluate \( f(4) \) if you only had a four-function calculator. First, you would put in 4. Then you would press \( \times 2 \), and then you would press \(+ 3\).

Notice that the order in which we do the operation is important. What would we have thought \( f(4) \) was if we accidentally added and then multiplied? You probably remember the order of operations from previous math classes. For review, the order is parentheses, exponents, multiplication and division, then addition and subtraction. You probably also remember the mnemonic “Please Excuse My Dear Aunt Sally”.

We can represent the actions we perform on the inputs of a function as a flowchart. The function \( f(x) = 2x + 3 \) could be represented as the flowchart

\[
\begin{align*}
&x \quad \text{multiply by} \quad 2 \quad \text{add} \quad 3 \quad f(x) \\
\end{align*}
\]

Here’s another example. Let

\[ g(x) = 5(x + 2)^2 + 3. \]

Then the flowchart of \( g(x) \) is

\[
\begin{align*}
&x \quad \text{add} \quad 2 \quad \text{square} \quad \text{multiply by} \quad 5 \quad \text{add} \quad 3 \quad g(x) \\
\end{align*}
\]
Let’s do one more example. Let

\[ h(x) = \frac{1}{2} \sqrt{4(x + 2) + 3} - 1. \]

The flowchart is

\[ \begin{array}{cccc}
  x & \rightarrow & \text{add} & \frac{1}{2} \\
  & & \rightarrow & \text{multiply by} 4 \\
  & & & \text{add} 3 \\
  & & & \rightarrow \text{square root} \\
  & & & h(x)
\end{array} \]

Let’s go backwards too. What is the function corresponding to the following flowchart?

\[ \begin{array}{cccc}
  x & \rightarrow & \text{multiply by} & \frac{-1}{2} \\
  & & \rightarrow & \text{add} 2 \\
  & & & \text{cube} \\
  & & & \rightarrow \text{add} -1 \\
  & & & \rightarrow \text{multiply by} 4 \end{array} \]

\[ f(x) \]

For the answer (don’t look there until after you try figuring it out), see the footnote\(^1\).

**Problem 1.** For each of the functions in problems 35–64 of section 3.5 in the book, draw the function flowcharts.

### 2 Basic Functions

**Problem 2.** Graph the following functions. We will be using them in the problems below.

1. \( x \)  
2. \( \frac{1}{x} \)  
3. \( x^2 \)  
4. \( \sqrt{x} \)  
5. \( x^3 \)  
6. \( \sqrt[3]{x} \)  
7. \( |x| \)  
8. \( \text{int}(x) \)

\(^1\)This was the function \( 4[(−x + 2)^3 − 1] \).
3 Translation

For the exercises in this section, use the function $f(x) = x^3$.

**Problem 3.** Write down the equation of the function having the flowchart

![Flowchart](image)

Graph the function. How is it different than $f(x)$? Be specific about the change, including whether the change was horizontal or vertical and how big the change was.

**Problem 4.** Repeat the previous exercise, but replace the “add 1” box with each of the following. Be sure to write down the equation, the graph, and how it is different from the graph of $f(x)$.

1. “add 2”
2. “add 3”
3. “add -1”
4. “add -2”
5. “add -3”

**Problem 5.** Write down the equation of the function having the flowchart

![Flowchart](image)

Graph the function. How is the graph different than the graph of $f(x)$? Be specific about the change, including whether the change was horizontal or vertical and how big the change was.

**Problem 6.** Repeat the previous exercise, but replace the “add 1” box with each of the following. Be sure to write down the equation, the graph, and how the graph is different from the graph of $f(x)$.

1. “add 2”
2. “add 3”
3. “add -1”
4. “add -2”
5. “add -3”

**Problem 7.** How did the graphs change when we put the “add” box at the beginning of the flowchart versus the end of the flowchart? (There are at least two major differences.)

### 4 Dilation

For the exercises in this section, use the function \( g(x) = \sqrt{x} \).

**Problem 8.** Write down the equation of the function having the flowchart

\[
x \rightarrow g(x) \rightarrow \text{multiply by 2} \rightarrow y
\]

Graph the function. How is the graph different than the graph of \( g(x) \)? Be specific about the change, including direction of change and how big the change was.

**Problem 9.** Repeat the previous exercise, but replace the “multiply by 2” box with the following. Be sure to write down the equation, the graph, and how the graph is different from the graph of \( g(x) \).

1. “multiply by 3”
2. “multiply by \( \frac{1}{2} \)”
3. “multiply by -1”
4. “multiply by -2”
5. “multiply by \( -\frac{1}{2} \)”

**Problem 10.** Write down the equation of the function having the flowchart

\[
x \rightarrow \text{multiply by 2} \rightarrow g(x) \rightarrow y
\]

Graph the function. How is the graph different than the graph of \( g(x) \)? Be specific about the change, including whether the change was horizontal or vertical and how big the change was.

**Problem 11.** Repeat the previous exercise, but replace the “multiply by 2” box with the following. Be sure to write down the equation, the graph, and how it is different from the graph of \( g(x) \).
1. “multiply by 3”
2. “multiply by \( \frac{1}{2} \)”
3. “multiply by -1”
4. “multiply by -2”
5. “multiply by -\( \frac{1}{2} \)”

Problem 12. How did the graphs change when we put the “multiply” box at the beginning of the flowchart versus the end of the flowchart? (There are at least two major differences.)

It is important to realize that dilations affect the whole plane and center around the \( x \)-axis (for horizontal dilations) or the \( y \)-axis (for vertical dilations). The dilations do not just stretch or compress the picture of the graph. For instance, where do you think the vertex of the parabola \( x^2 \) is after shifting up by 2 and then vertically stretching by \( \frac{1}{2} \) (i.e., compressing the function to half its height)? The answer is that the vertex is at \((0, 1)\), since every point, including \((0, 2)\), moves half the distance to the \( x \)-axis.

5 Change After Change

We will now investigate what happens when we do lots of changes to a graph. Earlier we briefly talked about why the order of translations and dilations is important. Are the graphs of \( 2x^2 + 1 \) and \( 2(x^2 + 1) \) different?

Problem 13.

(a) Draw the flowcharts of the functions \( f(x) = 2x^2 + 1 \) and \( g(x) = 2(x^2 + 1) \).
(b) How is the order of the boxes different in the flowcharts?
(c) Draw the graphs of \( f(x) \) and \( g(x) \).
(d) Draw the graph of \( x^2 \) that has been shifted up by 1 and then stretched vertically to twice its height.
(e) Draw the graph of \( x^2 \) that has been stretched vertically to twice its height and then shifted up by 1.
(f) Which function—\( f(x) \) or \( g(x) \)—corresponds to translating and then dilating \( x^2 \)?
(g) Which function—\( f(x) \) or \( g(x) \)—corresponds to dilating and then translating \( x^2 \)?
(h) Write down a rule which relates the order of boxes in the flowchart after the inside function and the order of vertical translations or dilations of the inside function.

**Problem 14.**

(a) Draw the flowcharts of the functions \( f(x) = (2x + 1)^2 \) and \( g(x) = (2(x + 1))^2 \).

(b) How is the order of the boxes different in the flowcharts?

(c) Draw the graphs of \( f(x) \) and \( g(x) \).

(d) Draw the graph of \( x^2 \) that has been shifted left by 1 and then compressed horizontally to half its width.

(e) Draw the graph of \( x^2 \) that has been compressed horizontally to half its width and then shifted left by 1.

(f) Which function—\( f(x) \) or \( g(x) \)—corresponds to translating and then dilating \( x^2 \)?

(g) Which function—\( f(x) \) or \( g(x) \)—corresponds to dilating and then translating \( x^2 \)?

(h) Write down a rule which relates the order of boxes in the flowchart before the inside function and the order of horizontal translations or dilations of the inside function.

### 6 Practice Makes Perfect

**Problem 15.** For problems 35–64 in the text, graph the functions by translating or dilating the inside function correctly. Draw each translation or dilation until you are completely comfortable with the whole process. After you are completely comfortable with how translations and dilations affect the graph (including multiple translations and dilations chained together), then just draw the final graph. Double check your answers with a calculator or computer. Be careful not to use the calculator as a crutch.