Chapter 1 Quiz

Please write your name in the upper right corner of this page. No calculators. Show your work.

1. Find the least upper bound of the following sets, if it exists.

   (a) $\left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z}, q \neq 0, p < 2 \right\}$ (Hint, what happens if $q = -1$ and p is a large negative number)
   Solution. If $q = -1$ and p is a large negative integer, then $\frac{p}{q}$ is a large number. We can make $\frac{p}{q}$ as large as we want this way. Therefore, there is no upper bound.

   (b) $\left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z}, q \neq 0, |p| < 2 \right\}$
   Solution. As opposed to the previous problem, since $|p| < 2$, then p can only be -1, 0, or 1. Therefore, the most $\frac{p}{q}$ can be is $\frac{1}{1} = 1$. Thus, the least upper bound is 1.

   (c) $(-\infty, 3)$
   Solution. 3.

   (d) $(3, \infty)$
   Solution. This set does not have a least upper bound.

   (e) $(0, 1] \cap [1, 2)$
   Solution. Since $(0, 1] \cap [1, 2) = \{1\}$, the least upper bound is 1.

2. Give the domain of each of the following functions.

   (a) $f(x) = \dfrac{1}{\sqrt{x+1}}$
   Solution. From the denominator, we need $x + 1 > 0$. Thus, the domain is $(-1, \infty)$.

   (b) $f(x) = \sqrt{x^2 - 1}$
   Solution. Since the domain of the square root is $[0, \infty)$, we need $x^2 - 1 \geq 0$. Thus, $x^2 \geq 1$, or $|x| \geq 1$. Thus, the domain is $(-\infty, -1] \cup [1, \infty)$.

   (c) $f(x) = \ln(x - 2)$
   Solution. Since the domain of the ln function is $(0, \infty)$, we need $x - 2 > 0$. Thus, the domain is $(2, \infty)$.

3. Find equations of the following lines.

   (a) The line perpendicular to the line $3y+2x = 4$ which passes through the point (0,2).
   Solution. The slope of the given line is $\frac{-2}{3}$, so the slope of the perpendicular line is $\frac{3}{2}$. Using this and the given point in the point-slope form of the line, we find the equation to be $y = \frac{3}{2}x+2$.

   (b) The line which passes through the points $(1, 0)$ and $(2, 3)$.
   Solution. Using the two-point form, we find the equation is $y = 3x - 3$.

4. The population of a certain colony was $1,000$ in 1870, and was $43,000$ in 1900. Assuming the population was growing exponentially, find an exponential function modeling the population as a function of time.
Solution. Let $f(x)$ be a function giving the population x years after 1870. Then $1000 = f(0) = Ab^0 = A$, so $A = 1000$. Using this and the other point, $43000 = f(30) = 1000b^{30}$. Therefore, $b = \sqrt[30]{43000} \approx 1.42708$. Thus, $f(x) = 1000\sqrt[30]{43000}^x = 1000(43000)^{x/30}$.

5. Find $\sin(\frac{\pi}{12})$ by writing $\frac{\pi}{12}$ in terms of $\frac{\pi}{3}$ and $\frac{\pi}{4}$.
Solution. Note that $\pi/12 = \pi/3 - \pi/4$. Using the sine subtraction formula,

$$\begin{aligned}
\sin(\pi/12) &= \sin(\pi/3 - \pi/4) \\
&= \sin(\pi/3)\cos(\pi/4) - \cos(\pi/3)\sin(\pi/4) \\
&= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}.
\end{aligned}$$

6. Find the coordinates of the maximum of the function

$$f(x) = -(2x-4)^2 + 3$$

Solution. There are two basic ways to do this. We can multiply out the quadratic, and notice that since it is a downward opening parabola, the vertex will be the maximum. Then, knowing that the vertex of a parabola is $\frac{-b}{2a}$, we can find the x coordinate of the maximum to be

2. The y coordinate is then found by plugging 2 to get 3. Thus, the coordinates are $(2,3)$.

The other way is to note that $-(2x-4)^2+3 = -(2(x-2))^2+3$, which is the x^2 function flipped upside down and translated so the vertex is at $(2,3)$. Thus, the coordinates of the maximum are $(2,3)$.

7. Find the inverse of $\dfrac{x}{3x-1}$.
Solution. Let $x = \frac{y}{3y-1}$ and solve for y to get $y = \frac{x}{3x-1}$. Thus, the inverse is the same as the original function.

8. Prove $1+\tan^2(x) = \sec^2(x)$.
Solution. We know that $\sin^2 x + \cos^2 x = 1$. Dividing both sides by $\cos^2 x$, we get $\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}$, which is $1+\tan^2 x = \sec^2 x$.

9. Find all real solutions to $\sin 2x = \cos x$ (Hint, use the double angle formula for sine to simplify).
Solution. Using the double angle formula ($\sin 2x = 2\sin x\cos x$), the equation simplifies to $2\sin x\cos x = \cos x$. Now, we have two cases: either $\cos x = 0$ or not.

$\cos x = 0$: If $\cos x = 0$, then $x = \pi/2 + 2k\pi$ or $x = 3\pi/2 + 2k\pi$, where $k \in \mathbb{Z}$. If $\cos x$ is 0, then both sides of the equation will be equal to 0, so these are solutions to the equation.

$\cos x \neq 0$: If $\cos x \neq 0$, then we can divide by $\cos x$ to simplify the equation to $2\sin x = 1$. Thus, we need to find solutions of $\sin x = 1/2$. The sine function is $1/2$ when $x = \pi/6 + 2k\pi$ or when $x = 5\pi/6 + 2k\pi$, where $k \in \mathbb{Z}$. Thus, these are other solutions to the equation.

Combining these solutions, we have $x = \frac{\pi}{2} + 2k\pi$, $x = \frac{3\pi}{2} + 2k\pi$, $x = \frac{\pi}{6} + 2k\pi$, or $x = \frac{5\pi}{6} + 2k\pi$, where $k \in \mathbb{Z}$.

10. Solve for x in each of the following equations.

    (a) $3^{-x}2^{x+1} = 5^x$
    Solution. There are many ways to solve this. Taking the logarithm of both sides, we see that $-x\ln 3 + (x+1)\ln 2 = x\ln 5$. Thus, $-x\ln 3 + x\ln 2 - x\ln 5 = -\ln 2$. Therefore $x = \frac{-\ln 2}{-\ln 3 + \ln 2 - \ln 5}$.

(b) $\log_2 2x + 2\log_2 x + 3 = \log_2 x$

Solution. Again, there are many ways to solve this. If we rewrite the equation as $\log_2 2 + \log_2 x + 2\log_2 x + 3 = \log_2 x$, and use $\log_2 2 = 1$, the equation simplifies to $\log_2 x = -4/2 = -2$. Converting to exponential notation, we have $2^{-2} = x$, which means $x = \frac{1}{4}$.

11. A certain radioactive substance decays from 35 grams to 20 grams in 8 days. What is its half-life?

Solution. We know $Q_0 = 35$. Using this and the second point, we have $20 = 35e^{-8k}$. Thus, $k = \frac{\ln(20/35)}{-8}$, and the function becomes $35e^{-\frac{\ln(20/35)}{-8}t}$. The half-life h will be when $Q = (1/2)Q_0$, so $\frac{1}{2} = e^{-kh}$ means $h = \frac{8\ln(1/2)}{\ln(20/35)}$. (Notice that h is still positive, because both logarithms are negative).

12. Find $\sin x$ if we know that $x = \cos^{-1}\frac{1}{3}$.

Solution. Drawing a triangle, we see that we are dealing with a right triangle with sides 1 and $\sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$ and hypotenuse 3. Therefore, $\sin x = \frac{\sqrt{8}}{3} = \frac{2}{3}\sqrt{2}$.