MATH 112 CHAPTER 1 SOLUTIONS

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Contents

The following are solutions to selected problems in Garner's *Calculus*.

1. Real Numbers

Exercise (3)**.** The complement of (a, b) in (c, d) is an interval when $a \leq c$ and $b > c$ or when $a < d$ and $b \geq d$.

Exercise (5)**.** *Unique Greatest Lower Bounds.* Let S be a set of real numbers that has a greatest lower bound. Let x and l be greatest lower bounds of S. Either $x < l$, $x > l$, or $x = l$.

- **$x < l$:** If $x < l$, then x cannot be a greatest lower bound because l is a greater lower bound. This is a contradiction.
- **$x > l$:** If $x > l$, then similarly, l cannot be a greatest lower bound because x is a greater lower bound. This is a contradiction.

Therefore $x = l$. Therefore, the greatest lower bound of S is unique. □

Exercise (7)**.** The answers are:

(a) $\text{glb}([0, 1]) = 0$, $\text{lub}([0, 1]) = 1$.
(b) $\text{glb}((0, 1)) = 0$, $\text{lub}((0, 1)) = 1$.
(c) $\text{glb}(\{1, 2, 3, \ldots\}) = 1$, $\text{lub}(\{1, 2, 3, \ldots\})$ does not exist.
(d) $\text{glb}(\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}) = 0$, $\text{lub}(\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}) = 1$.
(e) $\text{glb}((2, \infty)) = 2$, $\text{lub}((2, \infty))$ does not exist.

Exercise (A)**.** The answers are:

(a) $\text{lub}(\{x \in \mathbb{R} \mid x \geq 0 \text{ and } x^2 < 2\}) = \sqrt{2}$.
(b) $\text{lub}(\left\{\frac{2n}{n+1} \in \mathbb{R} \mid n \in \mathbb{N}\right\}) = 2$.

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(c) $\mathrm{lub}(\left\{\frac{p}{q} \in \mathbb{R} \mid p, q \in \mathbb{N}\right\})$ does not exist.

Exercise (B)**.** Let S be a nonempty, finite set of real numbers.

(a) Because S is finite, it has a greatest element and a smallest element. Let u be the greatest number and l be the smallest number in S. Then since no number in S is greater than u, and no number in S is smaller than l, S is bounded.
(b) Since $u \in S$, and u is an upper bound for S, $\mathrm{lub}(S) = u$.
(c) Since $l \in S$, and l is a lower bound for S, $\mathrm{glb}(S) = l$.

Exercise (9a)**.** Let $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n], \ldots$ be closed intervals such that $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$. Let $A = \{a_n \mid n \in \mathbb{N}\}$, the set of all left hand endpoints of the intervals. Then A is bounded above by b_1. Thus, by the Least Upper Bound Axiom, A has a least upper bound. Let $c = \mathrm{lub}(A)$.

Because $c = \mathrm{lub}(A)$, by definition $c \geq a_n$ for all $n \in \mathbb{N}$. Notice that for *every* $n \in \mathbb{N}$, A is bounded above by b_n. Since c is the *least* upper bound for A, we know $c \leq b_n$ for all $n \in \mathbb{N}$. Therefore, $a_n \leq c \leq b_n$ for all $n \in \mathbb{N}$. Therefore, $c \in [a_n, b_n]$ for every n.

Exercise (11)**.** To transition from step (1.3) to (1.4), we divide by 0 (since $a = b$). That is an illegal operation in this world, and leads to erronous results.

2. Functions, Graphs, and Modeling

Exercise (1)**.** (a) Graph II
(b) Graph III
(c) Graph I

Graph IV could describe "I started walking, but soon stopped to talk to a friend. After I finished talking, I continued walking."

Exercise (2)**.** See Figure 1.

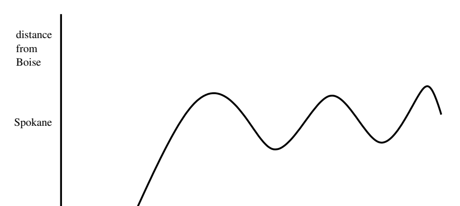

Figure 1. Exercise 2

Exercise (7)**.** See Figure 2

Exercise (9)**.** See Figure 3

Exercise (10)**.** See Figure 4. Note that the right hand endpoint of each line segment should be an open circle.

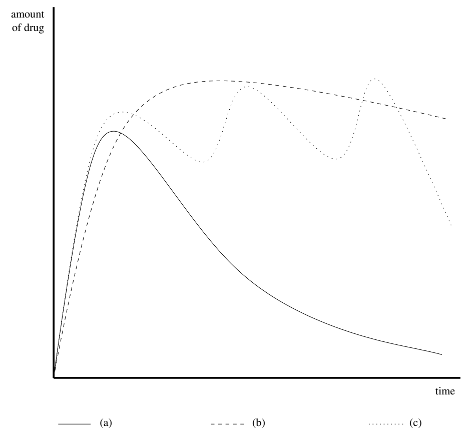

FIGURE 2. Exercise 7

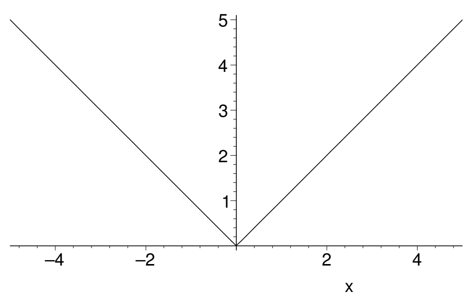

FIGURE 3. Exercise 9

Exercise (14)**.** See Figure 5

Exercise (27)**.** (a) $x \in [3, \infty)$
(b) $x \in (-\infty, -3] \cup [3, \infty)$
(d) $x \in (3, \infty)$
(f) $x \neq 4$

Exercise (28)**.** (a) $[0, \infty)$
(d) $(0, \infty)$

3. Linear Functions

Exercise (C)**.** To find the equation with slope -3 passing through the point $(1, 4)$, we use the slope-intercept form of the line:

$$\begin{aligned} f(x) &= 4 + (-3)(x - 1) \\ &= 4 - 3x + 3 \\ &= -3x + 7 \end{aligned}$$

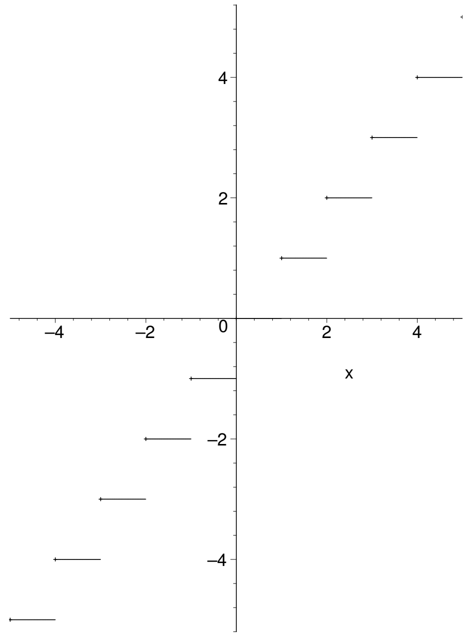

FIGURE 4. Exercise 10

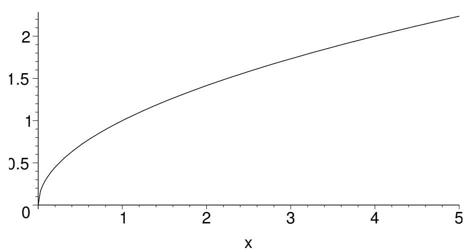

FIGURE 5. Exercise 14

Exercise (D)**.** To find the equation passing through the points $(3, 2)$ and $(5, 5)$, we use the two-point form of the line:

$$\begin{aligned} f(x) &= 2 + \frac{5-2}{5-3}(x-3) \\ &= 2 + \frac{3}{2}x - \frac{9}{2} \\ &= \frac{3}{2}x - \frac{5}{2} \end{aligned}$$

Exercise (2)**.**
(a) i
(b) vi
(c) iv
(d) ii
(e) v
(f) iii

Exercise (4)**.** Let $(a, 0) = (x_1, y_1)$ and $(0, b) = (x_2, y_2)$. Then plugging the points into the two-point form of the line, we get

$$y - 0 = \frac{b}{-a}(x - a).$$

Manipulating the equation, we get

$$\begin{aligned} -ay &= bx - ab \\ bx + ay &= ab \\ \frac{x}{a} + \frac{y}{b} &= 1. \end{aligned}$$

Exercise (5)**.** (a) Using the intercept form of the previous problem, we see that the equation is

$$\frac{x}{1} + \frac{y}{2} = 1.$$

Multiplying by 2 gives us the answer in the book, $2x + y = 2$.

Exercise (11)**.** Since the inclination of a line is determined by its slope ($\phi = \tan(m)$, where m is the slope and ϕ is the inclination), two lines will be parallel (have the same inclination) if and only if they have the same slopes.

Exercise (10)**.** Let $y = m_1 x + b_1$ and $y = m_2 x + b_2$ be two lines, with $m_1 \neq m_2$. By the answer above for Problem 11, the lines are not parallel because they have different slopes. Therefore, by the fundamental principles of Euclidean geometry, the lines intersect. To find the point of intersection, we just need to find a point (s, t) which satisfies both equations (and therefore will lie on both lines). From the first equation, $t = m_1 s + b_1$, and from the second, $t = m_2 s + b_2$. Since both equations are equal to t, we can set them equal and solve for s:

$$\begin{aligned} m_1 s + b_1 &= m_2 s + b_2 \\ m_1 s - m_2 s &= b_2 - b_1 \\ s(m_1 - m_2) &= b_2 - b_1 \\ s &= \frac{b_2 - b_1}{m_1 - m_2}. \end{aligned}$$

Then, substituting back into the first equation to find t,

$$\begin{aligned} t &= m_1 s + b_1 \\ &= m_1 \frac{b_2 - b_1}{m_1 - m_2} + b_1 \\ &= \frac{m_1 b_2 - m_1 b_1 + b_1(m_1 - m_2)}{m_1 - m_2} \\ &= \frac{m_1 b_2 - b_1 m_2}{m_1 - m_2}. \end{aligned}$$

Thus, the point $\left(\frac{b_2 - b_1}{m_1 - m_2}, \frac{m_1 b_2 - b_1 m_2}{m_1 - m_2}\right)$ is the point of intersection.

Exercise (16)**.** (a) See Table 1.

(b) The third column will probably be around 1.61803.

(c) If $F_{n+1}/F_n \to r$ and $F_{n+2}/F_{n+1} \to r$, then

$$r = \frac{F_{n+2}}{F_{n+1}} = \frac{F_n + F_{n+1}}{F_{n+1}} = \frac{F_n}{F_{n+1}} + 1 = \frac{1}{r} + 1$$

Index	Fib. Number	Ratio	Ratio
1	1	1	1.
2	1	2	2.
3	2	$\frac{3}{2}$	1.5
4	3	$\frac{5}{3}$	1.66667
5	5	$\frac{8}{5}$	1.6
6	8	$\frac{13}{8}$	1.625
7	13	$\frac{21}{13}$	1.61538
8	21	$\frac{34}{21}$	1.61905
9	34	$\frac{55}{34}$	1.61765
10	55	$\frac{89}{55}$	1.61818
11	89	$\frac{144}{89}$	1.61798
12	144	$\frac{233}{144}$	1.61806
13	233	$\frac{377}{233}$	1.61803
14	377	$\frac{610}{377}$	1.61804
15	610	$\frac{987}{610}$	1.61803
16	987	$\frac{1597}{987}$	1.61803
17	1597	$\frac{2584}{1597}$	1.61803
18	2584	$\frac{4181}{2584}$	1.61803
19	4181	$\frac{6765}{4181}$	1.61803
20	6765	$\frac{10946}{6765}$	1.61803
21	10946	$\frac{17711}{10946}$	1.61803
22	17711	$\frac{28657}{17711}$	1.61803
23	28657	$\frac{46368}{28657}$	1.61803
24	46368	$\frac{75025}{46368}$	1.61803
25	75025	$\frac{121393}{75025}$	1.61803

TABLE 1. Fibonacci Ratios

in the limit as $n \to \infty$. Thus, $r^2 = 1 + r$, which simplifies to $r^2 - r - 1 = 0$. Using the quadratic formula (and using the positive answer, since r is positive), we see that

$$r = \frac{1+\sqrt{5}}{2}.$$

This is the golden ratio, referred to in Problem 15, and is an important number in mathematics.

Exercise (23). (a) Using the two-point form, with the two points (x_1, y_1) and (x_2, y_2), we find the equation for the line between the two points is

$$g(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

(b) Using similar triangles is the same as evaluating $g(x)$ at c to find an estimate. Thus, the estimate will be

$$g(c) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(c - x_1).$$

(c) Errors can arise when the function $f(x)$ is not a linear function. For an example where $f(c) \neq g(c)$, see Figure 6.

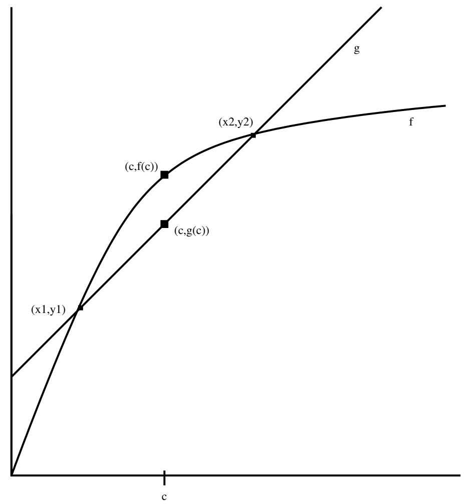

FIGURE 6. Exercise 23

4. Exponential Functions

Exercise (2). In general, a table for the exponential function looks like Table 2. To show we have a constant ratio between all consecutive entries in the table, we show there is a constant ratio between two arbitrary pairs of values in the table. The ratio of the values of the function for the first two entries is

$$\frac{Ab^{x_0}b^{\Delta x}}{Ab^{x_0}} = b^{\Delta x}.$$

Likewise the ratio between the second two entries is

$$\frac{Ab^{x_1}b^{\Delta x}}{Ab^{x_1}} = b^{\Delta x}.$$

Since the two ratios are equal, any two consecutive values in the table will have the same constant ratio, $b^{\Delta x}$.

A table for the linear function looks like Table 3. The difference between the values for two arbitrary consecutive entries is

$$m(x_0 + \Delta x) + b - (mx_0 + b) = mx_0 + m\Delta x + b - mx_0 - b = m\Delta x$$

and

$$m(x_1 + \Delta x) + b - (mx_1 + b) = mx_1 + m\Delta x + b - mx_1 - b = m\Delta x.$$

Since these differences are equal, any two consecutive values in the table will have the same constant difference, $m\Delta x$.

Note, we could have also concluded that since the ratio $b^{\Delta x}$ and the difference $m\Delta x$ do not depend on the starting point, x_0 or x_1, they are constant throughout the table.

Exercise (3). We take t to be the number of half-steps from the A below middle C. We will construct a function $f(t)$ which will give us the frequency of the note at t.

From the information given, we have the function values $f(0) = 220$ and $f(12) = 440$. From the first data point, we have $220 = f(0) = Ab^0 = A$. Therefore $A = 220$.

x	$f(x) = Ab^x$
$\vdots$	$\vdots$
x_0	Ab^{x_0}
$x_0 + \Delta x$	$Ab^{x_0+\Delta x}$
$\vdots$	$\vdots$
x_1	Ab^{x_1}
$x_1 + \Delta x$	$Ab^{x_1+\Delta x}$
$\vdots$	$\vdots$

TABLE 2. Table for $f(x) = Ab^x$

x	$f(x) = mx + b$
$\vdots$	$\vdots$
x_0	$mx_0 + b$
$x_0 + \Delta x$	$m(x_0 + \Delta x) + b$
$\vdots$	$\vdots$
x_1	$mx_1 + b$
$x_1 + \Delta x$	$m(x_1 + \Delta x) + b$
$\vdots$	$\vdots$

TABLE 3. Table for $f(x) = mx + b$

From the second data point, we have $440 = f(12) = 220b^{12}$. Therefore $b^{12} = 2$, and $b = 2^{1/12}$. Thus, the function is $f(x) = 220(2^{1/12})^t = 220(2^{t/12})$.

Exercise (4)**.** (a) Let t be the time in years from 1790. Let $f(t)$ calculate the population in the US, measured in millions. Then from the given data, we know $f(0) = 3.9$ and $f(10) = 5.3$. From the first data point, $3.9 = f(0) = Ab^0 = A$. Therefore, $A = 3.9$. From the second data point, $5.3 = f(10) = 3.9b^{10}$. Therefore $b^{10} = \frac{5.3}{3.9}$, so $b = \left(\frac{5.3}{3.9}\right)^{1/10}$. Therefore,

$$f(x) = 3.9\left(\frac{5.3}{3.9}\right)^{t/10} \approx 3.9(1.0311)^t.$$

(b) In 1860, the function predicts a population of $f(70) = 3.9\left(\frac{5.3}{3.9}\right)^{70/10} \approx 33.384$ million people. In 1870, the function predicts a population of $f(80) = 3.9\left(\frac{5.3}{3.9}\right)^{80/10} \approx 45.369$ million people. While the first prediction was fairly close, the second prediction did not take into account the Civil War, which drastically reduced the population in US.

Exercise (5)**.** Let t be the time in years, and $f(t)$ be the population at time t. Then from the given data, $f(0) = P_0$ and $f(1) = P_0(1.02)$. From the first point, $P_0 = f(0) = Ab^0 = A$. Therefore, $A = P_0$. From the second point, $P_0(1.02) = P_0b^1$. Therefore, $b = 1.02$. Thus, $f(x) = P_0(1.02)^t$.

To find the doubling time, we find what time t_{double} will give us $f(t_{\texttt{double}}) = 2P_0$. Thus, $P_0(1.02)^{t_{\text{double}}} = 2P_0$, and $(1.02)^{t_{\text{double}}} = 2$. We can estimate this on the calculator graphically, or solve exactly using logarithms to get $t_{\text{double}} \approx 35$ years.

Exercise (7). (a) We are given the points $f(0) = 2$ and $f(2) = 3$. From the first, $2 = f(0) = Ab^0 = A$. Using this and the second point, we see that $3 = f(2) = 2b^2$. Thus, $b = \sqrt{3/2} = (3/2)^{1/2}$. Thus, $f(x) = 2(3/2)^{t/2}$.

(c) We are given the points $f(0) = 1$ and $f(3) = 0.5$. From the first, $1 = f(0) = Ab^0 = A$. Using this and the second point, we see that $0.5 = f(3) = b^3$. Thus, $b = (0.5)^{1/3}$. Therefore, $f(x) = (0.5)^{t/3}$.

Exercise (8). See Figure 7.

Exercise (9). See Figure 8.

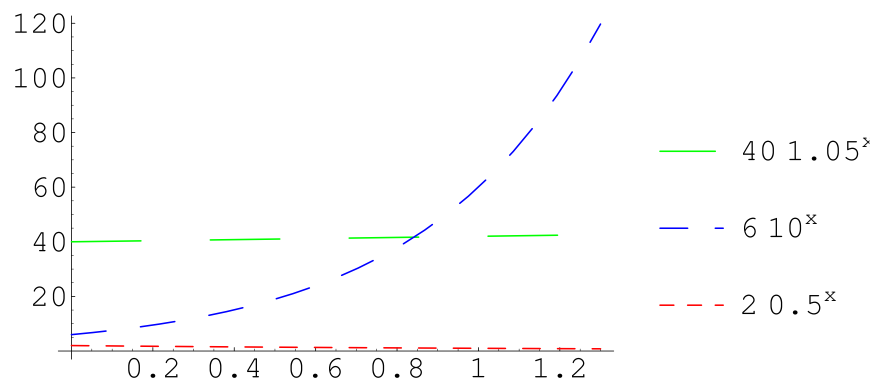

FIGURE 7. 1.4 #8

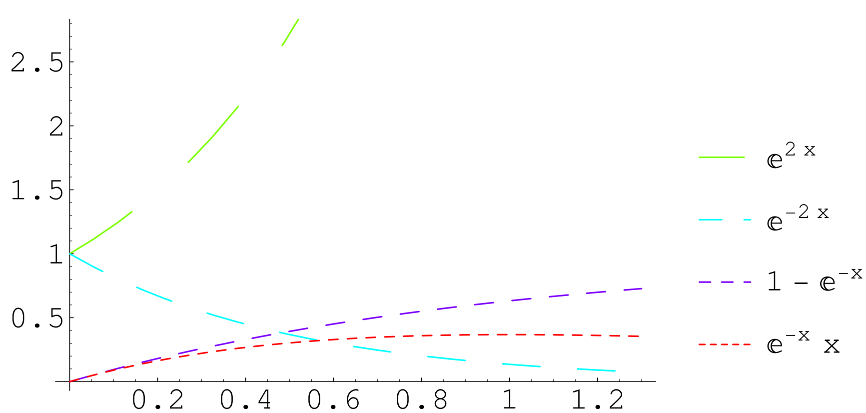

FIGURE 8. 1.4 #9

5. Circular Functions

Exercise (1). (a) 1 year.

(b) 27.3 days.

(d) 1 hour.

(e) 12 hours.

Exercise (2)**.** (b) Given that $\cos(\pi/2-t) = \sin(t)$, we can subsitute $t = \pi/2-x$ to get

$$\begin{aligned}\cos(\pi/2-(\pi/2-x)) &= \sin(\pi/2-x)\\ \cos(\pi/2-\pi/2+x) &= \sin(\pi/2-x)\\ \cos(x) &= \sin(\pi/2-x).\end{aligned}$$

(d) Using the cosine complement formula, we see that $\sin(s+t) = \cos(\pi/2-(s+t)) = \cos(\pi/2-s-t)$. We then simplify this expression using the cosine subtraction formula. Then we use both the cosine complement and sine complement formulas to get the expression we want.

$$\begin{aligned}\sin(s+t) &= \cos(\pi/2-s-t) = \cos((\pi/2-s)-t)\\ &= \cos(\pi/2-s)\cos(t)+\sin(\pi/2-s)\sin(t)\\ &= sin(s)\cos(t)+\cos(s)\sin(t)\end{aligned}$$

Exercise (3)**.** In each of the derivations below, we use the fact that $\cos(-x) = \cos(x)$ (because $\cos(x)$ is an even function), and $\sin(-x) = -\sin(x)$ (because $\sin(x)$ is an odd function. For the odd functions, we show $f(-x) = -f(x)$. For the even function, we show $f(-x) = f(x)$.

$\tan(x)$**:**

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x).$$

$\cot(x)$**:**

$$\cot(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos(x)}{-\sin(x)} = -\cot(x).$$

$\csc(x)$**:**

$$\csc(-x) = \frac{1}{\sin(-x)} = \frac{1}{-\sin(x)} = -\csc(x).$$

$\sec(x)$**:**

$$\sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos(x)} = \sec(x).$$

Exercise (4)**.** An isoceles right triangle will have two angles with radian measure $\pi/4$ (See Figure 9). If each leg is length a, then the hypotenuse will be length $a\sqrt{2}$ by the Pythagorean theorem. Using the triangle definitions for $\sin(x)$ and $\cos(x)$, we see that

$$\sin(\pi/4) = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

and

$$\cos(\pi/4) = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Exercise (6)**.** (a) Since March 21 is 80 days from January 1, the function gives $12.183+2.95\sin(0.0172(0)) = 12.183$ hours of daylight. We could also notice that on the equinox, we will be at the midpoint of our curve, so the sin part of the function must be zero.

(b) Since there are 172 days between January 1 and June 21, we evaluate $H(172)$ to get $H(172) = 12.183 + 2.95\sin(0.0172(172-80)) \approx 15.133$ hours of daylight. We could also observe that we will have the maximum amount of daylight on this day, and so the sin part of the function must be 1.

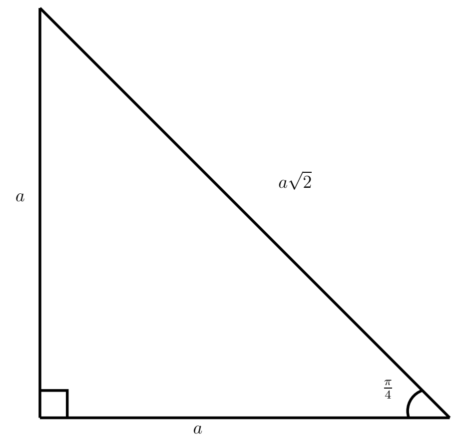

FIGURE 9. 1.5 #4

Exercise (8)**.** See Figures 10 and 11.

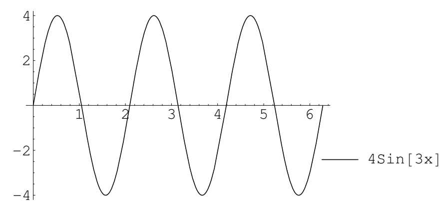

FIGURE 10. 1.5 #8a

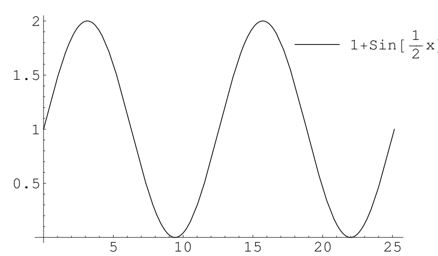

FIGURE 11. 1.5 #8k

Exercise (9)**.** (a) The curve has maximum at $x = 0$, so we start with a $\cos(x)$ curve. The period is about 2π, and the amplitude is 12. Therefore, $f(x) = 12\cos(x)$.

(b) The function is 0 at $x = 0$, so we start with a $\sin(x)$ curve. The amplitude is 4, and the period is about 2π. The equation is $f(x) = 4\sin(x)$.

(c) The curve appears to be a sin curve that has been shifted vertically up by one. The period appears to be about π, and the amplitude is 1. Therefore, $f(x) = \sin(2x) + 1$.

Exercise (13)**.** See Figures 12, 13, and 14.

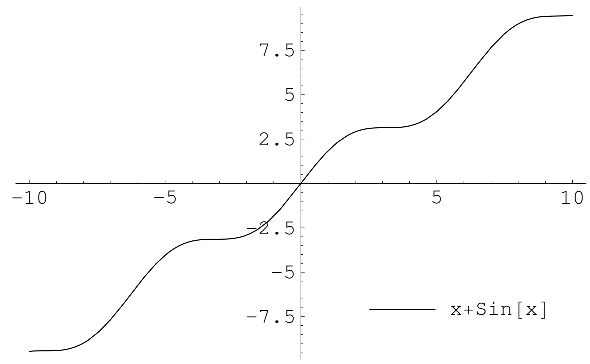

Figure 12. 1.5 #13a

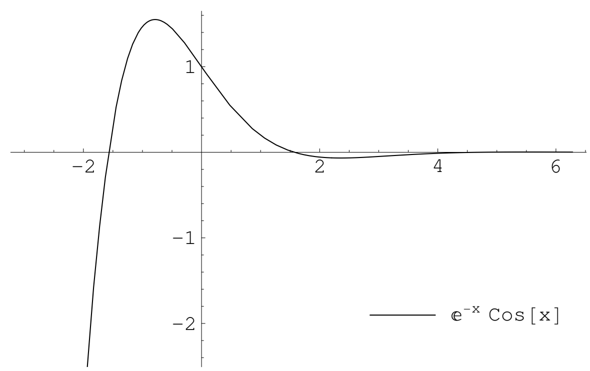

Figure 13. 1.5 #13f

6. New Functions From Old

Exercise (1)**.** (c)
- $(f+g)(x) = \cos(x) + x^3$.
- $(f-g)(x) = \cos(x) - x^3$.
- $(fg)(x) = \cos(x) \cdot x^3$.
- $(\frac{f}{g})(x) = \frac{\cos(x)}{x^3}$.
- $(g \circ f)(x) = \cos^3(x)$.
- $(f \circ g)(x) = \cos(x^3)$.

(f) First, we simplify $f(x)$ to $f(x) = \frac{x-1}{x} = -\frac{1-x}{x}$.
- $(f+g)(x) = \frac{x-1}{x} + \frac{1}{1-x}$.
- $(f-g)(x) = \frac{x-1}{x} - \frac{1}{1-x}$.
- $(fg)(x) = \frac{x-1}{x} \cdot \frac{1}{1-x} = -\frac{1}{x}$.
- $(\frac{f}{g})(x) = \frac{\frac{x-1}{x}}{\frac{1}{1-x}} = -\frac{(1-x)^2}{x}$.
- $(g \circ f)(x) = \frac{1}{1-\frac{x-1}{x}} = x$.

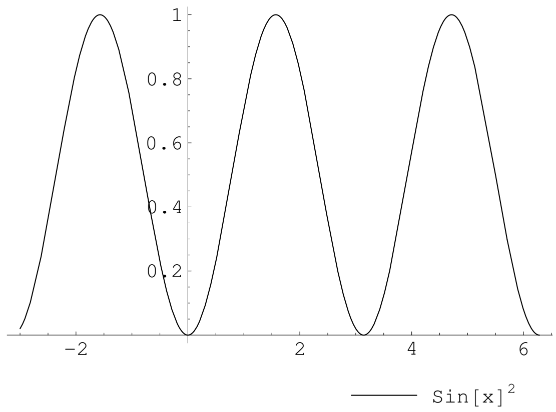

FIGURE 14. 1.5 #13g

- $(f \circ g)(x) = \frac{\frac{1}{1-x} - 1}{\frac{1}{1-x}} = x.$

Exercise (5)**.** The graph $h(t)$ is a downward opening parabola since the coefficient of t^2 is negative. The maximum of $h(t)$ is then the vertex of the parabola. This occurs when $t = \frac{-b}{2a} = \frac{v_0}{g}$. The height at this time is

$$h\left(\frac{v_0}{g}\right) = s_0 + v_0\left(\frac{v_0}{g}\right) - \left(\frac{g v_0^2}{2g^2}\right) = s_0 + \frac{v_0^2}{g} - \frac{v_0^2}{2g} = s_0 + \frac{v_0^2}{2g}.$$

Exercise (9)**.** Since 4 is a zero, $(x-4)$ is a factor. Using synthetic or long division, we find that $f(x) = (x-4)(x^2 - 2x - 2)$. Using the quadratic formula, we see that the zeros of the quadratic are $1 \pm \sqrt{3}$.

Exercise (16)**.** Since the windows are the same size, there is no compression or expansion of the x or y axis. If the new window is centered at $(2, 3)$, then it will appear as if the original graph has been translated to the right two units and down 3 units. Thus, we would have to plot $f(x+2) - 3$ in the original window to have a graph that looks like $f(x)$ in the new window.

Exercise (20)**.** The window (a) is approximately $[0, 4] \times [0, 4]$. Window (b) is approximately $[0, 100] \times [0, 10000]$. Window (c) is approximately $[0, 200] \times [0, 10000000]$.

7. INVERSE FUNCTIONS

Exercise (1)**.**
(a) $f(x) = 2(x+3)$. $f^{-1}(x) = x/2 - 3$. The domain of f^{-1} is all real numbers.
(c) $f(x) = x^3 + 1$. $f^{-1}(x) = (x-1)^{1/3}$. The domain of f^{-1} is all real numbers.
(h) $f(x) = (x+2)^4$. This function is not $1-1$ (for example, $f(-1) = f(-3) = 1$).

Exercise (2)**.**
(a) f is $1-1$. $f^{-1}(x) = (x-2)^{1/3}$.
(d) f is not $1-1$ (for example, $f(1) = f(-1) = 3$).
(e) f is $1-1$. $f^{-1}(x) = \frac{x+1}{x-1}$.

Exercise (5)**.** See Figure 15.

Exercise (6)**.** See Figure 16.

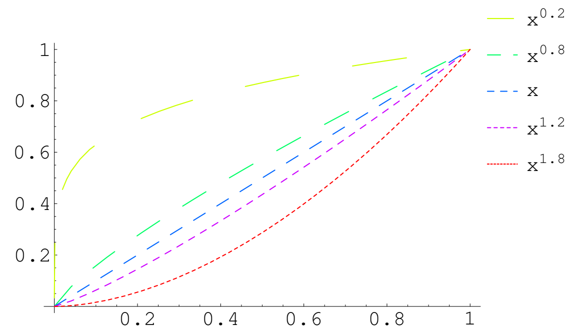

FIGURE 15. 1.7 #5

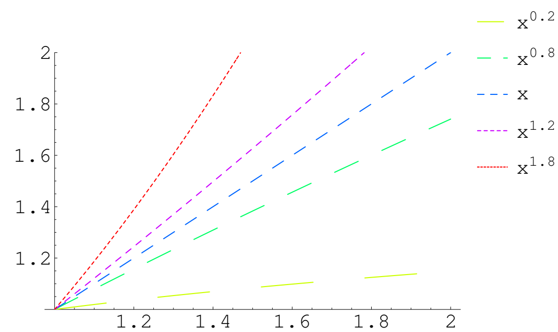

FIGURE 16. 1.7 #6

Exercise (8)**.** Let $f(x) = mx + b$ be a linear function. Since the function is $1 - 1$, it has an inverse function. Switching the x and y values and solving for y, we find the inverse.

$$x = my + b$$
$$\therefore y = \frac{x - b}{m}$$

Thus, $f^{-1}(x) = (1/m)(x - b) = (1/m)x - b/m$, which is a linear function.

8. Logatithmic functions

Exercise (1)**.** (a) $\log_2(16x) = \log_2 2^4 + \log_2 x = 4 + \log_2 x$.
(c)) $\ln(4ab^2c^3 = \ln 4 + \ln a + \ln b^2 + \ln c^3 = \ln 4 + \ln a + 2\ln b + 3\ln c$.
(f) $\ln((1 + x)e^{-3x}) = \ln(1 + x) + \ln e^{-3x} = \ln(x + 1) - 3x$.

Exercise (3)**.** (a) Since $3^x = 5$, we take logarithms of both sides to obtain $\ln 3^x = \ln 5$. Thus, $x \ln 3 = \ln 5$, and $x = \frac{\ln 5}{\ln 3}$.

(c)

$$\begin{aligned}
3 \cdot 2^{x-1} &= 2 \cdot 3^{x-2} \\
\therefore \ln 3 + \ln 2^{x-1} &= \ln 2 + \ln 3^{x-2} \\
\therefore \ln 3 + (x-1)\ln 2 &= \ln 2 + (x-2)\ln 3 \\
\\
\therefore x \ln 2 - x \ln 3 &= \ln 2 - \ln 3 + \ln 2 - 2\ln 3 \\
\therefore x(\ln 2 - \ln 3) &= 2\ln 2 - 3\ln 3 \\
\therefore x &= \frac{\ln \frac{4}{27}}{\ln \frac{2}{3}}.
\end{aligned}$$

(f)

$$\begin{aligned}
1 + \ln(3x) &= 3 + \ln(x^2) \\
\therefore \ln(3x) - \ln(x^2) &= 2 \\
\therefore \ln\left(\frac{3x}{x^2}\right) &= 2 \\
\therefore \ln\left(\frac{3}{x}\right) &= 2 \\
\therefore e^2 &= \frac{3}{x} \\
\therefore x &= \frac{3}{e^2}
\end{aligned}$$

Exercise (5)**.** Let $f(t) = Q_0 e^{-kt}$ be the general exponential decay function, where $Q_0 = f(0)$. The half-life h is the time in which the function becomes $f(h) = Q_0/2$. Using this data point, we see that

$$\begin{aligned}
\frac{Q_0}{2} &= Q_0 e^{-kh} \\
\therefore \frac{1}{2} &= e^{-kh} \\
\therefore \ln \frac{1}{2} &= -kh \\
\therefore -\ln 2 &= -kh \\
\therefore \ln 2 &= kh.
\end{aligned}$$

Exercise (8)**.** Let $f(t) = Q_0 e^{-kt}$ be the exponential decay function in the problem, measuring the grams of substance as a function of days. Then from the information given, $f(0) = 12$ and $f(12) = 5$. From the first point, $Q_0 = 12$. Using this and the second point, $5 = f(12) = 12e^{-12k}$. Therefore, $\ln \frac{5}{12} = -12k$. Therefore, $k = \frac{\ln \frac{5}{12}}{-12}$. (Note that this is a positive number, since $\ln \frac{5}{12} < 0$). Using the results from Problem 5, $h = \frac{\ln 2}{k} = \frac{-12 \ln 2}{\ln \frac{5}{12}}$. (Again, note that h is positive).

Exercise (10)**.** Using the compound interest formula $A = Pe^{rt}$ (which really just the generic exponential growth formula with $k = r$ and $Q_0 = P$), we see that we will have $1000e^{06(10)} \approx 1822.12$. (Note that the correct answer is given in the errata for the book).

Exercise (14)**.** Using the results of Problem 5, and the given fact that C-14 has a half life of 5730 years, the exponential decay function of C-14 has a decay rate $k = \frac{\ln 2}{5730} \approx .00012097$. Thus, the function modeling the amount of C-14 in a substance after t years is $f(t) = Q_0 e^{-\frac{\ln 2}{5730}t}$, where Q_0 is the amount of $C-14$ when the substance died.

(a) From the data in the problem, $(.64)Q_0 = Q_0 e^{-kt}$. Thus, $.64 = e^{-kt}$. Therefore, $t = \frac{\ln .64}{-k}$. Using the above calculated value for k, we see that $t \approx 3689$ years.
(b) Using the above formula and value for k, $f(2615) = Q_0 e^{-k(2615)} \approx Q_0(.729)$. Therefore, about 72.9% of the original C-14 should remain.
(c) From the data in the problem, $(.91)Q_0 = Q_0 e^{-kt}$. Thus, $.91 = e^{-kt}$. Therefore, $t = \frac{\ln .91}{-k} \approx 780$ years. So according to the test, the shroud is from around $1988 - 780 = 1200$ A.D.

Exercise (22)**.** Let $f(t) = Q_0 e^{kt}$ be the exponential function modeling the number of bacteria as a function of hours. Then from the data, $f(4) = 5600$ and $f(6) = 6800$. Thus, $Q_0 e^{4k} = 5600$ and $Q_0 e^{6k} = 6800$. Dividing these equations, we get

$$\begin{aligned} \frac{6800}{5600} &= \frac{Q_0 e^{6k}}{Q_0 e^{4k}} \\ \frac{17}{14} &= e^{2k} \\ \ln \frac{17}{14} &= 2k \\ k &= \frac{\ln \frac{17}{14}}{2} \approx 0.097078. \end{aligned}$$

Now, using this value of k and the first data point, we see that

$$\begin{aligned} 5600 &= Q_0 e^{\frac{\ln \frac{17}{14}}{2}(4)} \\ Q_0 &= \frac{5600}{e^{\frac{\ln \frac{17}{14}}{2}(4)}} \approx 3800. \end{aligned}$$

Thus, there were approximately 3800 bacteria at time $t = 0$.

Exercise (28)**.** There are several ways to do this. One way is to use $\log_b\left(\frac{y}{x}\right) = \log_b y - \log_b x$ and $\log_b 1 = 0$, and let $y = 1$. Using these, $\log_b \frac{1}{x} = \log_b 1 - \log_b x = 0 - \log_b x = -\log_b x$.

Exercise (29)**.** (c) Using the change of base formula, $\log_2 5 = \frac{\ln 5}{\ln 2} = \frac{\log_{10} 5}{\log_{10} 2}$. Take your pick as to which to evaluate, and you find $\log_2 5 \approx 2.32193$.

9. Other Elementary Functions

Exercise (1)**.** (a) $\frac{\pi}{6}$.
(b) $\frac{\pi}{6}$.
(c) $-\frac{\pi}{6}$ (don't forget the range of $\arcsin x$ is $[-\pi/2, \pi/2]$.
(d) $-\frac{5\pi}{6}$ (don't forget the range of $\arccos x$ is $[0, \pi]$.
(h) $\cosh 0 = \frac{1}{2}(e^0 + e^0) = 1$.
(j) $\sinh 1 = \frac{1}{2}(e^1 - \frac{1}{e}) = \frac{e^2-1}{2e}$.

Exercise (4)**.** (b) If $\cos(x) = 0.5$, then x is either $\pi/3 + 2\pi k$ or $-\pi/3 + 2\pi k$, where $k \in \mathbb{Z}$. Another way to write this is $2\pi k \pm (\pi/3)$.

Exercise (6). (a) If $\sin x = 1$, then $x = \pi/2 + 2k\pi$, where $k \in \mathbb{Z}$.

(b) If $\sin 2x = 1$, then $2x = \pi/2 + 2k\pi$, where $k \in \mathbb{Z}$. Thus, $x = \pi/4 + k\pi$, where $k \in \mathbb{Z}$.

(d) If $\sin x \cos x = \frac{1}{3}$, then $2 \sin x \cos x = \frac{2}{3}$. Using the double angle formula ($\sin 2x = 2 \sin x \cos x$), we see that $\sin 2x = \frac{2}{3}$. Thus, $2x$ is either $\arcsin \frac{2}{3} + 2k\pi$ or $\pi - \arcsin \frac{2}{3} + 2k\pi$, where $k \in \mathbb{Z}$. Thus, x is either $\frac{1}{2} \arcsin \frac{2}{3} + k\pi$ or $\frac{\pi}{2} - \frac{1}{2} \arcsin \frac{2}{3} + k\pi$. Another way to write this is $\frac{n\pi}{2} + (-1)^n \frac{1}{2} \arcsin \frac{2}{3}$.

Exercise (9). (b) $-\frac{5\pi}{6}$.

(c) $\operatorname{arcsec}(\csc \frac{\pi}{3}) = \operatorname{arcsec} \frac{2}{\sqrt{3}} = \frac{\pi}{6}$.

Exercise (11). Use a triangle to solve for the remaining values.

(a) $\sin(x) = 1/2, \cos(x) = \sqrt{3}/2, \tan(x) = 1/\sqrt{3}$.

(c) $\sin(x) = 3/\sqrt{34}, \cos(x) = 5/\sqrt{34}, \tan(x) = 3/4$.

(e) $\sin(x) = a/\sqrt{a^2+1}, \cos(x) = 1/\sqrt{a^2+1}, \tan(x) = a$.

Exercise (15). (a) Using the definition of $\cosh(x)$ and $\sinh(x)$, we work out the algebra:

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \frac{1}{4}((e^x + e^{-x})^2 - (e^x - e^{-x})^2) \\
&= \frac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}) \\
&= \frac{1}{4}(4) \\
&= 1
\end{aligned}$$

Exercise (17). Let $y = \sinh^{-1} x$. Then $\sinh y = x$. Using the definition of sinh, we find that

$$\begin{aligned}
2x &= e^y - e^{-y} \\
&= \frac{e^{2y} - 1}{e^y} \\
\therefore e^{2y} - 2xe^y - 1 &= 0
\end{aligned}$$

Now, if we consider $(e^y)^2 - 2x(e^y) - 1$ as a quadratic equation, we can solve for e^y using the quadratic formula. We get $e^y = x + \sqrt{x^2+1}$. Then, taking the natural log of both sides, we get the desired answer.