MATH 112 SOLUTIONS FOR SECTION 2.3, P. 132

1. (b) 1. (d) DNE. (f) $\frac{3}{4}$. (h) DNE. (j) -4. (k) $\frac{3}{2}$.
2. (b) $-\frac{1}{2}$. (d) DNE. (e) 0. (f) $\frac{1}{4\sqrt{2}}$. (g) $\frac{1}{2\sqrt{2}}$. (h) $\frac{1}{6}$. (k) 1. (l) $\frac{2}{3}$.

E. If we substitute in $x = 1$ in the top and the bottom, we get the $0/0$ case. So we look for a factor of $x - 1$ in the top and in the bottom. Since $x^2 - 1 = (x+1)(x-1)$ and $x^3 - 1 = (x-1)(x^2+x+1)$ (use polynomial or synthetic division to divide x^3-1 by $x-1$), we can cancel the $x-1$ from the top and from the bottom. Thus, $\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{x+1}{x^2+x+1} = \frac{2}{3}$.

3. (a) $\sqrt{2}$. (d) $\frac{2}{\sqrt{3}}$. (e) $\frac{2}{3}$. (f) 2. (g) $\frac{2}{3}$. (h) (l) DNE. $\frac{2}{3}$. (k) 0.
4. (a) 1. (b) 0. (d) 0. (e) 0. (h) DNE. (i) 0. (j) DNE (undefined for $x > \pi$).
5. Theorem. $\lim\limits_{x\to c} f(x) = L \iff \lim\limits_{x\to c} \big||f(x) - L|\big| = 0$.

   Proof ($\Rightarrow$) If $f(x) \to L$ as $x \to c$, then $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon \Rightarrow ||f(x) - L| - 0| < \epsilon$. Hence $|f(x) - L| \to 0$ as $x \to c$.

   ($\Leftarrow$) If $|f(x)-L| \to 0$ as $x \to c$, then $0 < |x-c| < \delta \Rightarrow ||f(x)-L|-0| < \epsilon \Rightarrow |f(x) - L| < \epsilon$. Therefore $f(x) \to L$ as $x \to c$.
6. For example, $f(x) = \dfrac{|x|}{x}$ has no limit as $x \to 0$, but $\lim_{x\to 0} \left|\dfrac{|x|}{x}\right| = 1$.
8. Theorem. If $\lim\limits_{x\to c} f(x) = L \neq 0$ and $\lim\limits_{x\to c} g(x) = 0$, then $\lim\limits_{x\to c} f(x)g(x) = 0$.

   Proof. Since $f(x) \to L$ as $x \to c$, we can choose x close enough to c that $|f(x)| < L + 1$. Then, given any $\epsilon > 0$, we can choose x close enought to c that $|g(x)| < \frac{\epsilon}{L+1}$. Then, with x close enough to c, we have $|f(x)g(x) - 0| = |f(x)||g(x)| < (L + 1)\cdot\frac{\epsilon}{L+1} = \epsilon$.
14. (a) 0 (b) 0 (c) 0

    (d) Undetermined. For example, if $f(x) = \sin x$ and $g(x) = \sin^2 ax$, then $\lim_{x\to 0} \dfrac{g(x)}{xf(x)} = a^2$, which depends on a.

    (e) Undetermined. For example, if $g(x) = \sin^2 ax$, then $\lim_{x\to 0} \dfrac{g(x)}{x^2} = a^2$, which depends on a.
16. $\sin x = \sin(\frac{x-c}{2} + \frac{x+c}{2}) = \sin\frac{x-c}{2}\cos\frac{x+c}{2} + \cos\frac{x-c}{2}\sin\frac{x+c}{2}$, and
    $\sin c = \sin(\frac{c-x}{2} + \frac{c+x}{2}) = -\sin\frac{x-c}{2}\cos\frac{x+c}{2} + \cos\frac{x-c}{2}\sin\frac{x+c}{2}$. Hence
    $\sin x - \sin c = 2\sin\frac{x-c}{2}\cos\frac{x+c}{2}$.
18. (a) bounded, $B = 1$. (b) not bounded. (c) bounded, $B = 1.56$. (d) not bounded. (e) bounded, $B = 1$. (f) bounded, $B = 1$.
19. $f(x) = x$ is not bounded, but $\frac{f(x)}{x} = 1$ is bounded for x not 0. If $\lim_{x\to 0} f(x)$ exists and is some nonzero number, then $\frac{f(x)}{x}$ is not bounded near 0. Hence if $\frac{f(x)}{x}$ is bounded for all x, then either $f(x) \to 0$ as $x \to 0$ or $\lim_{x\to 0} f(x)$ does not exist.

20. (a) $s = 2r\sin\frac{\pi}{n}$. (b) $P = 2rn\sin\frac{\pi}{n}$. (c) $C = \lim_{n\to\infty} 2rn\sin\frac{\pi}{n} = 2\pi r \lim_{n\to\infty}\dfrac{\sin\pi/n}{\pi/n} = 2\pi r\cdot 1 = 2\pi r$. (d) The area of an inscribed polygon with n sides is $A_n = nr^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ and the area of the circle is $A = \lim_{n\to\infty} A_n = \pi r^2\lim_{n\to\infty}\frac{\sin\pi/n}{\pi/n}\cos\frac{\pi}{n} = \pi r^2$.