MATH 112 SOLUTIONS FOR SECTION 2.5, P. 154

The following solutions do not include the graphs. You will need to include the appropriate graphs in your solutions you hand in.

1. (a) Define $f(2) = 4$. (b) Let $f(3) = 6$. (f) Redefine $f(2) = \frac{1}{2}$.
2. (a) $f(x) \to -1$ as $x \to 0-$ and $f(x) \to 1$ as $x \to 0+$.
   (c) $f(x) \to 2$ as $x \to 1-$ and $f(x) \to 0$ as $x \to 1+$.
3. (a) $1/x^2 \to \infty$ as $x \to 0$. (c) $e^{-1/x} \to \infty$ as $x \to 0-$.
4. (a) $f(x) = \frac{x+3}{x-3}$ has an infinite discontinuity at 3 (b) $f(x) = \frac{x+3}{x-3}$ is continuous at -3. (c) continuous.
5. (b) continuous except at $x = -1$ (c) continuous except at $x = 0$ (e) $-2 \leq x \leq 2$.
6. (b) not continuous at 1. (c) yes. (e) infinite discontinuity at 0. (f) yes. (g) yes.
7. Let $L(t)$ be the number of liters of fuel in the tank and $D(t)$ the distance traveled. Let $f(t) = L(t) - D(t)$. Clearly, $f(0) > 0$ and if T is when $L(T) = 0$, then $f(T) < 0$. If f is continuous, there is a point t_0 such that $f(t_0) = 0$, which means that $L(t_0) = D(t_0)$.

14. Correction: In line 2, remove "> 0".

    (a) Correction: Change "B" to "C". Solution: Let $C = \max\{|b|, |B|\}$. Then $|f(x)| \leq C$.

    (b) Let f be continuous on an interval (a, b) containing c, and let $\epsilon > 0$ be given. Then since f is continuous at c, there is $\delta > 0$ such that if $|x - c| < delta$ and x is in the domain of f, then $|f(x) - f(c)| < \epsilon$. Thus if $c - \delta < x < c + \delta$, then $f(c) - \epsilon < f(x) < f(c) + \epsilon$, so f is bounded below by $f(c) - \epsilon$ and above by $f(c) + \epsilon$ over $(c - \delta, c + \delta)$.

    (c) If f is bounded on $[a, b]$, there exists $B > 0$ such that $|f(x)| \leq B$ for $x \in [a, b]$. If f is bounded on $[b, c]$, then these exists $C > 0$ such that $|f(x)| \leq C$ for $x \in [b, c]$. L:et $D = \max\{B, C\}$. Then for $x \in [a, c]$, $|f(x)| \leq D$ and f is bounded on $[a, c]$.

    (d) Theorem. If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.

    Proof. Let $S = \{x \in [a, b] \mid f \text{ is bounded on } [a, x]\}$. Since f is defined at a, $a \in S$ and S is non-empty. If $x \in S$, then $x \leq b$, so that S is bounded above. Let c be the least upper bound of S. Then $c \leq b$; suppose $c < b$. Then given any $\epsilon > 0$ there exists $\delta > 0$ such that $c + \delta < b$ and $|f(x) - f(c)| < \epsilon$. That means that $|f(x)| < |f(c)| + \epsilon$ for $x \in (c, c + \delta)$, so that f is bounded beyond c, and c is not the least upper bound of S after all. This contradiction forces $c = b$, and thus f is bounded on $[a, b]$.

15. (a) Since f is continuous on $[a, b]$, f is bounded on $[a, b]$. That is, R_f is bounded above, so has a least upper bound; call it M.

    (b) If $g(x) = \dfrac{1}{M - f(x)}$, then $g(x) > 0$. If $f(x)$ is always less than M, then $g(x)$ is continuous on $[a, b]$, and hence bounded on $[a, b]$.

    (c) Let $B > 0$ and $\epsilon = \frac{1}{B}$. Because M is the least upper bound of R_f, $M - \epsilon$ is not an upper bound of R_f.

(d) There is some $x \in [a, b]$ such that $f(x) > M - \epsilon$. Then $\epsilon > M - f(x) \Rightarrow B < \dfrac{1}{M - f(x)} = g(x)$.

(e) Since B is arbitrary, $g(x)$ must not be bounded. Hence $f(c) = M$ for some $c \in [a, b]$, and f achieves a maximum on $[a, b]$.

(f) Let $F(x) - f(x)$. Then F is continuous on $[a, b]$, so by (a)-(e) F achieves a maximum at some point $c \in [a, b]$. That is, $F(c) \geq F(x)$ for allk $x \in [a, b]$. Hence $f(c) = -F(c) \leq -F(x) = f(x)$ for all $x \in [a, b]$, and f achieves a minimum at c.

16. Suppose to the contrary there are two numbers, c_1 and c_2, such that $a_n \leq c_1, c_2 \leq b_n$ for every n. Then $|c_1 - c_2| < b_n - a_n \to 0$ means $c_1 = c_2$. Hence c is unique.

17. Zero Theorem. If f is continuous on $[a, b]$, $f(a) < 0$, and $f(b) > 0$, then there is some $c \in [a, b]$ such that $f(c) = 0$.

    Proof. Let $c_1 = \dfrac{a+b}{2}$. If $f(c_1) = 0$, we are done. If $f(c_1) < 0$, let $a_1 = c_1$ and $b_1 = b$. If $f(c_1) > 0$, let $a_1 = a$ and $b_1 = c_1$. Then $[a_1, b_1]$ is a subinterval of $[a, b]$, f is continuous on $[a_1, b_1]$, $f(a_1) < 0$, and $f(b_1) > 0$. Let $c_2 = \dfrac{a_1 + b_1}{2}$. If $f(c_2) = 0$, we are done. If $f(c_2) < 0$, let $a_2 = c_2$ and $b_2 = b_1$; if $f(c_2) > 0$, let $a_2 = a_1$ and $b_2 = c_2$. Then $[a_2, b_2]$ is a subinterval of $[a_1, b_1]$. Continue in this fashion. We thus either find a point c_n such that $f(c_n) = 0$, and are done, or we create a sequence of intervals $\{[a_n, b_n]\}$ such that $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$. By the Nested Interval Theorem, there is a point c such that $c \in [a_n, b_n]$ for all n. Since c_{n+1} is the midpoint of $[a_n, b_n]$, the length of $[a_n, b_n]$ is $\dfrac{b-a}{2^n}$, which approaches 0 as $n \to \infty$. Thus c is unique. Now suppose $f(c) > 0$. Let $\epsilon = \frac{1}{2} f(c)$. Since f is continuous at c, there is some $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for $|x - c| < \delta$. But $\dfrac{b-a}{2^n} < \delta$ for some n, making $c - a_n < \delta$, so that $|f(a_n) - f(c)| < \epsilon = \frac{1}{2} f(c) \Rightarrow -\frac{f(c)}{2} < f(a_n) - f(c) < \frac{f(c)}{2} \Rightarrow \frac{f(c)}{2} < f(a) < \frac{3f(c)}{2}$. But $f(a_n) < 0$, a contradiction. Hence it cannot be that $f(c) > 0$. Similarly, it cannot be that $f(c) < 0$, so $f(c) = 0$.

18. Let f be continuous on $[a, b]$ and let v be a value between $f(a)$ and $f(b)$. First suppose that $f(a) < f(b)$. Then let $g(x) = f(x) - v$. Then $g(x)$ is continuous on $[a, b]$, $g(a) < 0$, and $g(b) > 0$. Hence by the Zero Theorem there is a point c such that $g(c) = 0 = f(c) - v$, and $f(c) = v$. If $f(a) > f(b)$, let $g(x) = v - f(x)$; the same result follows.

24. Given any interval containing a point c, there will be values of f differing by 1 at points in that interval. Thus given any $\epsilon > 0$, if $\epsilon < 1$ then it will be impossible to force $|f(x) - f(c)| < \epsilon$ by choosing x close to c.

31. (a) $p(0) = 4, p(-1) = -1$, so p has a zero between 0 and –1.
    (b) $p(-2) = 12, p(-1) = -1$, so p has a zero between –2 and –1.
    (a) $p(1) = 3, p(2) = -04$, so p has a zero between 1 and 2.

32. (b) Given $f(x) = x - \cos x$ on [0, 1], we find $f(0) < 0$ and $f(1) > 0$, so f, being continuous, will have a zero between 0 and 1. Four bisections yield the interval [.6875, .75].