MATH 112 SOLUTIONS FOR 5.1, P. 359

1. (a) $\{1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\}$. (b) $LHS(4) = \frac{11}{4}, RHS(4) = \frac{13}{4}$. (c) $LHS(8) = \frac{23}{8}, RHS(8) = \frac{25}{8}$. (d) $\{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \cdots, 1+\frac{n-1}{n}, 2\}$. $LHS(n) = 2 + \dfrac{2}{n^2}[1+2+3+\cdots+(n-1)], RHS(n) = 2 + \dfrac{2}{n^2}[1+2+\cdots+n]$. (e) $LHS(n) = 3 - \frac{1}{n}, RHS(n) = 3 + \frac{1}{n}$. (f) 3 (g) The trapezoid has bases 2 and 4 and altitude 1, so has area 3.
2. (a) π. (b) $LHS(n)$ will be an overstimate, $RHS(n)$, an underestimate, because function is decreasing. (c) 3.58422; 2.25089. (d) 3.39532; 2.72865. (e) Yes; $LHS(n)$ should decrease as n increases, and $RHS(n)$ should increase.
3. (a) Because the function is increasing, $LHS(n)$ is an underestimate and $RHS(n)$ is an overestimate. (b) $LHS(3) = 0.715249$, $RHS(3) = 1.238848$. (c) $LHS(6) = 0.863382$, $RHS(6) = 1.125182$. (d) As n increases, underestimates should increase and overestimates should decrease. That is what we see here. (e) about 1.
4. $A \approx \frac{32}{3}$.
5. (a) There is no inscribed polygon, so the inner area is zero. (b) The smallest circumscribed polygon is a square with area 1, so the outer area is 1.
7. Proof: Suppose that $\mathbb{N} - S$ is non-empty. Then $\mathbb{N} - S$ has a least element m. Now $1 \in S$, so $m \neq 1$. Hence $m - 1 \in S$, since m is the least element not in S, so $m = (m-1) + 1 \in S$, a contradiction. Therefore $S = \mathbb{N}$.
8. (a) Let $S = \{n \in \mathbb{N} \mid 1 + 2 + \cdots + n = \frac{n(n+1)}{2}\}$. $1 \in S$ since $1 = \frac{1(1+1)}{2}$. Suppose $k \in S$. Then $1+2+\cdots+k = \frac{k(k+1)}{2}$. Hence $1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2|} + (k+1) = \frac{(k+1)(k+1+1)}{2}$, and $k+1 \in S$. Hence $S = \mathbb{N}$, and the given statement is true for all positive integers n.

   (b) Let $S = \{n \in \mathbb{N} \mid 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}\}$. $1 \in S$ because $1 = \frac{(1)(1+1)(2\cdot 1+1)}{6}$. Suppose $k \in S$. Then $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(2k+1)+g(k+1)}{6} \cdot (k+1) = \frac{(k+1)(2k^2+7K+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$, and $k+1 \in S$. Hence $S = \mathbb{N}$, and the statement is true for all positive integers n.

   (e) Let $S = \{n \in \mathbb{N} \mid 8^n - 3^n \text{ is divisible by } 5\}$. Then $8^1 - 3^1 = 5$ is divisible by 5, so $1 \in S$. Suppose $k \in S$. Then $8^k - 3^k$ is divisible by 5. Hence $8^{k+1} - 3^{k+1} = 8^{k+1} - 3 \cdot 8^k + 3 \cdot 8^k - 3^{k+1} = 8^k(8-3) + 3(8^k - 3^k)$ is divisible by 5 because each term is. Hence $k+1 \in S$, so $S = \mathbb{N}$ and $8^n - 3^n$ is divisible by 5 for every positive integer n.
10. If $\delta = \frac{\epsilon}{4}$, then $|x_1 - x_2| < \delta \Rightarrow |x_1^2 - x_2^2| = |x_1 - x_2||x_1 + x_2| < \frac{\epsilon}{4} \cdot 4 = \epsilon$, because $x_1, x_2 < 2$.
11. (a) If f is uniformly continuous, then given $\epsilon > 0$ there exists $\delta > 0$ such that if $p_1, p_2 \in [a, b]$ and $|p_1 - p_2| < \delta$ then $|f(p_1) - f(p_2)| < \frac{\epsilon}{b-a}$, by the definition in Problem 10.

    (b) If $\frac{b-a}{n} < \delta$, then let M_i and m_i be the points in the ith subinterval at which f has its maximum and minimum, respectively. Then $f(M_i) -$

$f(m_i) < \frac{\epsilon}{b-a}$, by part (a), so that

$$\begin{aligned} US(n) - LS(n) &= [f(M_1) + \cdots + f(M_n)]\frac{b-a}{n} - [f(m_1) + \cdots + f(m_n)]\frac{b-a}{n} \\ &= [(f(M_1) - f(m_1)) + \cdots (f(M_n) - f(m_n)]\frac{b-a}{n} \\ &< [\frac{\epsilon}{b-a} + \cdots \frac{\epsilon}{b-a}]\frac{b-a}{n} \\ &= n \cdot \frac{\epsilon}{b-a} \cdot \frac{b-a}{n} \\ &= \epsilon. \end{aligned}$$

(c) If $US(n) - LS(n) < \epsilon$, then as $n \to \infty$, $US(n) - LS(n) \to 0$. That means $US(n)$ and $LS(n)$ have the same limit, so R has area.