MATH 112, SECTION 9, WINTER 2003
TEST 5, MARCH 20-22
NO CALCULATORS.

INSTRUCTOR: JASON GROUT

ABSTRACT. Fifth test retesting material in sections 3.1-3.9.

PART I. MULTIPLE CHOICE

Instructions: Circle the best answer.

1. (5pts) If f is twice differentiable in the interval (0,2), $f'(1) = 0$, and $f''(1) = 3$, then which of the following can we conclude.
   (a) f has a local minimum at $x = 1$.
   (b) f has a local maximum at $x = 1$.
   (c) f has an inflection point at $x = 1$
   (d) We can conclude none of the above.
   Solution. The second derivative test says that f has a local minimum at $x = 1$.
2. (5pts) Let $f(x) = x^3 - 3x^2 - 9x + 4$. Which of the following is true?
   (a) f is concave down on $(-1, 3)$.
   (b) f is concave up on $(1, \infty)$.
   (c) f is increasing on $(-1, 3)$.
   (d) f is decreasing on $(1, \infty)$.
   (e) None of the above.
   Solution. $f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3). f''(x) = 6x - 6 = 6(x-1)$. It follows that (b) is correct
3. (5pts) Find $\dfrac{dy}{dt}$ when $x = 0$, $y = 1$, $\dfrac{dx}{dt} = 2$, and

$$ye^{\tan(x)} - 3x = 4y$$

   (a) $-3/4$
   (b) $-3/2$
   (c) -1
   (d) -2
   (e) None of the above.
   Solution. The answer is $-4/3$, which corresponds to (e).
4. (5pts) Use differentials to approximate $\sqrt{66}$.
   (a) $8\frac{1}{8}$
   (b) $8\frac{1}{9}$

(c) $9\frac{1}{8}$
(d) $8\frac{1}{10}$
(e) $8\frac{1}{4}$
(f) $8\frac{3}{16}$
(g) None of the above
Solution. (a) $\sqrt{66} \approx 8 + \frac{1}{2} \cdot \frac{1}{8} \cdot 2$.

Part II Short Answer

Instructions: Show your work. Please write your solutions in the space provided. Clearly mark your answers.

1. (5pts) A block of ice shaped as a cube is melting. Using linear approximation, estimate the change in volume of the ice as the side length changes from 5 inches to 4.99 inches.
   Solution. $V = s^3$. $dV = 3s^2 ds$. So $dV \approx 3(5^2)(.01) = 3/4$
2. (5pts) Ben called his family in Idaho Falls at noon to tell them that he was coming home for the weekend. The distance to Idaho Falls from Provo is approximately 280 miles. At 3:30 Ben knocked on the door to his family's house. Use the Mean Value Theorem to prove that at some point on his trip, Ben was traveling faster than 75 mi/hr; i.e he was speeding.
   Solution. We assume that Ben's position function $s(t)$ of distance from Prove is continuous. The mean value theorem say that there exists a $c \in (0, 3.5)$ with $v(c) = s'(c) = \dfrac{s(3.5) - s(0)}{3.5 - 0} = \dfrac{280}{7/2} = 80$. Hence at some point on the trip, Ben was going 80 miles an hour.
3. (5pts) Find the x-intercept of the tangent line to the curve $f(x) = x^5 - 4x + 2$ at $x = 1$.
   Solution. $x_2 = -1$
4. (5pts) Find the critical point(s) of f. Assume that a and b are positive constants
   $$f(x) = axe^{-bx}$$
   Solution. $x = \frac{1}{b}$
5. (8pts) Find the absolute maximum and absolute minimum of $f(x) = x^3 - 3x + 1$ on the interval [0,2]. (Remember that the absolute max and min are y values)
   Solution. $f'(x) = 0$ at $x = \pm 1$. Since -1 is not in the domain, we ignore it. The critical point are thus $0, 1, 3$. The corresponding y values are $1, -1, 3$. Abs max is 3; Abs min is -1
6. Find the following limits: (5pts each)
   (a) $\lim\limits_{x \to 0} \dfrac{x^5 + 3x}{e^{2x} - 1}$
   Solution. $\frac{3}{2}$. Applying L'Hôpital's rule once gives us $\lim_{x \to 0} \frac{5x^4 + 3}{2e^{2x}} = 3/2$
   (b) $\lim\limits_{x \to \infty} (1 + x)^{1/x}$
   Solution. $e^0 = 1$

(c) $\lim_{x\to\infty} \sqrt{x^2-8x}-x$

Solution. Multiply by the conjugate in the numerator and denominator, and then divide the top and bottom by x. You get -4.

7. (8pts) A camera is located 50 ft from a straight road along which a car is traveling at 100 ft/sec. The camera turns so that it is pointed at the car at all times. In radians/sec, how fast is the camera turning as the car passes closest to the camera? ***Solution.*** The setup gives $\tan\theta = \frac{x}{50}$. Taking derivatives with respect to time we get

$$\begin{aligned}\frac{d\theta}{dt}\sec^2\theta &= \frac{1}{50}\frac{dx}{dt}\\ \frac{d\theta}{dt}\sec^2 0 &= \frac{1}{50}100\\ \frac{d\theta}{dt}1 &= 2\\ \frac{d\theta}{dt} &= 2\end{aligned}$$

we get $\frac{d\theta}{dt} = 2$

$2r$

h 8

8. (8pts) A stranded resident of a desert island wants to fit a small cylindrical vial inside a hollow spherical coconut shell having an inside diameter of eight inches. Find the volume of the largest vial that can fit inside the shell. The volume of a cylinder equals $\pi r^2 h$ where h is the height and r is the radius.

Solution. First, $h^2+(2r)^2 = 8^2$. We find that the critical points occur when $h = \sqrt{\frac{64}{3}}$. You easily note that this gives the maximum, so the maximum volume is $\pi\left(\frac{64-64/3}{4}\right)\sqrt{\frac{64}{3}} = \frac{256\pi}{3\sqrt{3}}$.

9. (8pts) Suppose the cost of producing q units of a certain item is given by $C(q) = 40 - 50q - .5q^2$. Suppose that the demand for the item is given by $q(p) = 80 - p$. Find the price you should charge to maximize profit.

Solution. You should pay people \$50 to take your products to maximize your profit ($p = -50$).

10. (12pts) Let $f(x) = \dfrac{2x}{x^2+1}$. Give the following information, and sketch the graph of f.

Vertical asymptotes	__________
Horizontal asymptotes	__________
Intercepts	__________
Local maxima	__________
Local minima	__________
Inflection points	__________
Intervals on which $f(x)$ is:	
Increasing	__________
Decreasing	__________
Concave up	__________
Concave down	__________

Solution.

Vertical asymptotes	No vertical asymptotes.
End behavior	$y \to 0$ as $x \to \pm\infty$
Intercepts	$(0, 0)$
Local maxima	$x = 1$ ($y = 1$)
Local minima	$x = -1$ ($y = -1$)
Inflection points	$x = -\sqrt{3}, 0, \sqrt{3}$
Intervals on which $f(x)$ is:	
Increasing	$(-1, 1)$
Decreasing	$(-\infty, -1) \cup (1, \infty)$
Concave up	$(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$
Concave down	$(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

The graph is shown in Figure 1.

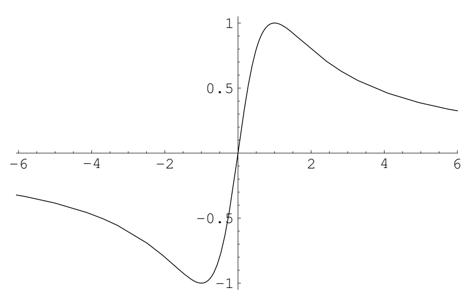

FIGURE 1. Problem 10