Abstract. Fourth test covering sections 4.1-4.3.

Part I. Multiple Choice

Instructions: Circle the best answer.

1. (5 pts) Which of the following is an antiderivative of \( \frac{1}{3 - x} \)?
   (a) \( \ln |3 - x| \)
   (b) \( \ln |3 - x| + 5 \)
   (c) Both of the above.
   (d) None of the above.

   Solution. (d). Both of the first answers are missing the negative sign on the \( \ln \).

2. (5 pts) Let \( C \) be a constant. The general antiderivative of \( \frac{1}{\sqrt{1 - x^2}} \) is
   (a) \( \tan^{-1} x + C \)
   (b) \( \sec^{-1} x + C \)
   (c) \( \sin^{-1} x + C \)
   (d) None of the above.

   Solution. (c).

3. (5 pts) Which of the following are solutions to the differential equation
   \[ \frac{d^2s}{dt^2} = -\omega^2 s \]
   (a) \( s = 3 \cos(\omega t) \)
   (b) \( s = 5 \sin(\omega t) \)
   (c) \( s = 3 \cos(\omega t) + 5 \sin(\omega t) \)
   (d) All of the above.
   (e) None of the above.

   Solution. (d).
Part II. Short Answer

Instructions: Show your work. Clearly mark your answers. No calculators.

1. (5 pts each) Find the general antiderivatives of the following functions. Let $k$ be a constant.
   (a) $e^{-kx}$ Solution. $\frac{-ke^{-kx}}{k} + C$
   (b) $\sin kx$ Solution. $\frac{-\cos kx}{k} + C$
   (c) $k \cosh x$ Solution. $k \sinh x + C$
   (d) $3x^{-2} + 5x^{-1} + 7x^2$ Solution. $-3x^{-1} + 5 \ln x + \frac{7x^3}{3} + C$

2. (8 pts each) Solve the following differential equations, given the boundary conditions. Assume that $y$ is a function of $x$.
   (a) $2y = \csc^2 x$, $y\left(\frac{\pi}{2}\right) = 1$. Solution. $y = \sqrt{-\cot x + 1}$
   (b) $y' + 1 = -y'x^2$, $y(1) = \frac{\pi}{4}$. Solution. $y = -\arctan x + \frac{\pi}{2}$
   (c) $y' - 3 = -4y$, $y(0) = 0$. Solution. $y = \frac{3e^{-4x} - 3}{-4}$

3. (5 pts) Prove that $\frac{a^{bx}}{b \ln a} + C$ is the general antiderivative of $a^{bx}$. Solution.
   The derivative of $\frac{a^{bx}}{b \ln a} + C$ is $\frac{a^{bx}(\ln a)(b)}{b \ln a} = a^{bx}$. Therefore, the antiderivative of $a^{bx}$ is as stated.

4. (5 pts) A projectile is thrown downward from the top of a building at a speed of 3 ft/s. If the projectile hits the ground 2 seconds later, how high is the building? Assume the only acceleration acting on the projectile is $g = 32 \text{ ft/s}^2$, the constant of gravity. Solution. 70 ft

5. (10 pts) A mass $Q$ of a radioactive substance decays at a rate proportional to the mass of the substance.
   (a) Set up a differential equation describing the change in mass of the substance with respect to time. Solution. $\frac{dQ}{dt} = kQ$, where $k$ is a constant.
   (b) A certain lump of the substance decays from 27 grams to 9 grams in 1 day. Solve the differential equation modeling the situation and find when there will be only 1 gram remaining. Solution. At the end of 3 days.

6. (10 pts) A certain plant grows in height at a rate inversely proportional to the square root of its height.
   (a) Set up a differential equation describing the change in height with respect to time. Solution. $\frac{dH}{dt} = \frac{k}{\sqrt{H}}$, where $k$ is a constant.
   (b) It takes 8 years for the plant to reach a height of 4 meters. Solve the differential equation and find how tall the plant will be when it is 27 years old. Solution. 9 meters

7. (10 pts) The temperature $Q$ of a cup of water is found to change with respect to time at a rate proportional to the difference between its temperature and $A$, the temperature of the surrounding air.
(a) Set up a differential equation describing the rate of change of temperature of the cup of water with respect to time.  

**Solution.**  \( \frac{dQ}{dt} = k(Q - A) \), where \( k \) is a constant.

(b) Jason places a cup of water with temperature 10 degrees Celsius in a freezer that is kept at -10 degrees Celsius. One hour later, he notes that the temperature of the cup is 5 degrees Celsius. Solve the differential equation modeling the situation to find a function \( Q(t) \) that gives the temperature of the cup at the time \( t \). Prove your solution is correct.

**Solution.**  \( Q(t) = -10 + 20e^{\ln(3/4)t} \). Using this, we see that the cup freezes at \( t = \frac{\ln(1/2)}{\ln(3/4)} \) hours.