Exam 5 — Math 112 — Fall 2004

Write the best answer in the box provided. For multiple choice and true/false questions, put your answer in the blank provided. Show your work.

1. $\int f(x) \, dx > 0$, then $f(x) > 0$ for each $x$ in $[a, b]$.
   (6 pts) T/F

2. $\int$ (6 pts) Suppose $F(x)$ and $G(x)$ are both antiderivatives of a continuous function $f(x)$.
   If $F(3) - G(3) = 4$, then find $F(0)$ if $G(0) = 2$.
   
   (a) -2
   (b) -1
   (c) 0
   (d) 2
   (e) 5
   (f) 6

3. (6 pts) Suppose that $f$ is continuous and positive at all real numbers. Let
   
   $$F(x) = \int_0^x f(t) \, dt.$$ 
   
   Then $F$ is
   
   (a) increasing in some places, decreasing in others
   (b) increasing
   (c) decreasing
   (d) None of the above can be concluded.

4. $\int$ (6 pts) The following figure shows the inflow and outflow rates of a certain reservoir, where $t$ is measured in months, and $y$-axis represents thousands of gallons per month.
   
   Which of the following best describes what happens to the reservoir in the 3 months shown?

   (a) The amount of water decreased by about 1000 gallons.
   (b) The amount of water decreased by about 3000 gallons.
   (c) The amount of water decreased by about 6000 gallons.
   (d) The amount of water increased by about 1000 gallons.
   (e) The amount of water increased by about 3000 gallons.
   (f) The amount of water increased by about 6000 gallons.
   (g) The water level is about the same as what it was 3 months ago.
5. (6 pts each) Find the following, using any techniques we have learned.

(a) $\int_{-2}^{3} |x| \, dx$

(b) $\int_{1}^{e} \frac{\ln x}{x} \, dx$

(c) $\int_{-3}^{3} \frac{x^2 \tan x + 2 \, dx}{\cos x}$

(d) $\int 2x \sqrt{x} + 3 \, dx$

(e) $\int xe^{-x^2/3} \, dx$

6. (6 pts) Compute $\frac{d}{dx} \int_{x}^{\sin x} e^{\sqrt{t}} \, dt$.
7. (18 pts) This problem tests your understanding of the numerical approximations found in the chapter for estimating definite integrals. Use \( f(x) = x^3 \), for \( x \in [-1, 3] \).

(a) (3 pts) Compute \( L_f(2) \).

(b) (3 pts) Compute \( R_f(2) \).

(c) (3 pts) Compute \( T_f(2) \).

(d) (3 pts) Compute \( M_f(2) \).

(e) (3 pts) Compute \( SR_f(4) \).

\[ \frac{1}{3} \left( T_f(2) + 2M_f(2) \right) = 20 \]

(f) (3 pts) Compute \( \int_{-1}^{3} f(x) \, dx \).

\[ \int_{-1}^{3} x^3 \, dx = 20 \]

Note that this is a check, since Simpson's rule is exact on \( x^3 \).

8. (6 pts) Find all points of inflection of the function \( A(x) = \int_0^x \frac{1}{4+x^2} \, dt \).

\[ A'(x) = \frac{1}{4+x^2} \]

\[ A''(x) = -\frac{1}{(4+x^2)^2} \left( 2x \right) = 0 \]

\( x = 0 \)
9. (10 pts) The goal of this problem is to find the area of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

\( (a) \) (3 pts) Compute \( \int_0^1 \sqrt{1-u^2} \, du \).

\( \frac{\pi}{4} \)

\( (b) \) (4 pts) Compute \( \int_0^a b \sqrt{1 - \left( \frac{x}{a} \right)^2} \, dx \), by finding an appropriate substitution.

\[ u = \frac{x}{a} \quad \text{and} \quad du = \frac{1}{a} \, dx \]

\[ ab \int_0^1 \sqrt{1-u^2} \, du = a b \frac{\pi}{4} \]

\( (c) \) (3 pts) How does the integral from (b) relate to the area of the ellipse? Find the area of the ellipse.

Check this if \( a = b = r \), then we have a circle of radius \( r \), and the area is \( \pi r^2 \).

10. (6 pts) Parley has an ant farm that is 2 feet wide. Before making any tunnels, the industrious ants express their love for calculus by piling up the dirt so that the top of the dirt takes the shape of the curve \( y = 1 - x + x^2 \). Since Parley prefers more interesting curves, he shakes the ant farm until the dirt lies flat. What is the height of the dirt now?

We are looking for the average value

\[ \int_0^2 (1-x+x^2) \, dx = \left[ x - \frac{x^2}{2} + \frac{x^3}{3} \right]_0^2 \]

\[ = 2 - 2 + \frac{8}{3} = \frac{8}{3} \]

\[ \text{The average value is } \frac{1}{2} \int_0^2 (1-x+x^2) \, dx = \frac{1}{2} \cdot \frac{8}{3} = \frac{8}{6} = \frac{4}{3} \]

\[ \frac{4}{3} \text{ ft.} \]