Mathematics of Investments

You want to retire in 40 years. You figure you’ll have to open some kind of savings account, so you scrape up your savings and look at your options.

Exercise 1. One bank offers you a 1% annual interest rate (which is really good these days), compounded annually. This means that the bank will pay you 1% of your money at the end of the year.
(a) If you initially deposit $1,000, how much will you have in one year?
(b) How much will you have in two years? (don’t forget that you will earn interest on the money the bank paid you for the first year).
(c) How much will you have in three years? Five years?
(d) How much will you have in n years? (write a formula)
(e) How much will you have in 40 years?
(f) If you manage to deposit $10,000, how much will you have in 40 years?

Exercise 2. Another bank offers you a 1% annual interest rate, but this time it is compounded quarterly. This means the bank will pay you every quarter. Since they are paying four times as often, they pay you \(\frac{1}{4}\) of your interest rate each time, so the bank pays you \((\frac{1}{4})(1\%) = 0.25\)% of your money each time. Assume again that you deposit $1,000.
(a) How much money will you have in three months (1 quarter)?
(b) Six months (2 quarters)?
(c) One year? Two, three, and five years?
(d) How much will you have in n years? (write a formula)
(e) How much will you have in 40 years?
(f) If you manage to deposit $10,000, how much will you have in 40 years?

Exercise 3 (Compounding again). It seems that compounding more often results in the bank paying you more. Suppose you have $1,000 to invest at 1% annual interest.
(a) Suppose a bank compounds every day (365 times a year). How much will you have at the end of a year?
(b) How much will you have in a year if a bank compounds every hour (365*24=8,760 times a year)?
(c) How much will you have in a year if a bank compounds every minute (365*24*60=525,600 times a year)?

The amounts you have at the end of the year seem to approach a certain number. To find out what that number is, we rewrite the compounding formula as

\[ F = P \left( 1 + \frac{1}{n/r} \right)^{n/r} \]

and we examine the quantity \((1 + \frac{1}{n/r})^{n/r}\) as \(n\) becomes larger and larger. If we fix \(r\), then \(n/r\) becomes large as \(n\) becomes large. Let \(m = n/r\).

Exercise 4 (The number e). Fill in the table for the values of \((1 + \frac{1}{m})^m\).

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<thead>
<tr>
<th>(m)</th>
<th>((1 + \frac{1}{m})^m)</th>
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<tr>
<td>1</td>
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These values seem to approach a number as \(n\) gets bigger and bigger. We call this number \(e\), and an approximation is \(e = 2.718281828459045\).

Returning to our original problem, if we compound continuously (i.e., \(n\) is infinite), then

\[ F = Pe^{rt}. \]

Suppose you invest $1,000 and a bank will pay you 1% interest, compounded continuously.
(a) How much money will you have after 1 year?
(b) How much money will you have after 5 years?
(c) How much money will you have after 40 years?
(d) How much money will you have if you invest $10,000 for 40 years?

You quickly realize that you need to save up quite a bit more money to be able to retire and live on your savings for several decades.
Exercise 5 (Inflation). Living costs increase in the next 40 years as well. A standard rule of thumb is that living expenses will increase at 3% each year. If you plan on having a retirement income of $24,000 (in today’s living costs), how much will you need in 40 years? Hint: How much will $24,000 grow, at 3% per year, in 40 years?

Exercise 6 (Present Value).
(a) Assume that you will live for 30 years after you retire. If you need $80,000 a year for 30 years, how much will you need saved up when you start your retirement in 40 years? Don’t account for inflation after you retire. When you retire, you put your money into an account that earns interest at the rate of inflation.
(b) The value of your investment in the future is called the future value of your investment. The value of your investment right now is called the present value. Assume that a bank will pay you 1% interest, compounded continuously. If you want to have $2,400,000 in 40 years, how much do you need to invest right now? In other words, what is the present value of $2,400,000 in 40 years at 1% interest compounded continuously?
(c) That seemed like a lot. A good money market fund will pay about 2% right now. Assume that you invest in a money market fund. How much do you need to invest now to have a future value of $2,400,000 in 40 years?
(d) Even that seems like quite a bit. A 10 year CD has interest rate of 3.50%. How much do you need to invest in that now?
(e) The average return on a good growth mutual fund over a long period of time is about 10%. How much do you need to invest to have $2,400,000 in 40 years?

Exercise 7 (How fast will it grow?). Suppose you invest in a fund that yields 10% annual interest.
(a) You would first like to see how much your money grows when it is invested now compared to several years from now. How much will $1 grow with 10% interest for 40 years? (i.e., how much will every dollar you invest be worth?)
(b) How much will that same $1 grow if invested next year (so it only grows for 39 years)?
(c) How much will the $1 earn if invested in 5 years? (i.e., it earns interest for 35 years?)
(d) How much will the $1 earn if invested in 10 years?
(e) How much will the $1 earn if invested in 20 years?

Exercise 8 (The Early Bird). From the last problem, we see that money we invest early will grow drastically more than money we invest later. You decide to take advantage of that knowledge and plan to invest early. You still can’t invest all the money right now, though. Working out an investment plan is one way to gradually add to your retirement fund. Suppose you invest in a mutual fund that pays you 10% interest (compounded continuously).
(a) You decide to invest for the next 5 years, but then you figure that your expenses will not allow you to invest again. If you invest $2,000 a year for the next 5 years, how much will you have in 40 years? Hint: Calculate the future value of each of the $2,000 yearly investments and add them.
(b) That’s not quite enough. Suppose you invest $3,000 a year for the next 5 years. How much will you have in 40 years?
(c) That’s still not enough. Suppose you invest $6,000 a year for the next 5 years. How much will you have in 40 years?
(d) We’re getting closer to the amount we need in 40 years. Suppose you refigure your finances and calculate that you can invest $6,000 a year for the next 8 years.
(e) We’re getting tired of writing out all these calculations. Suppose you can invest for the next 10 years. Write a formula for how much you will have in 40 years if you invest n dollars each year for 10 years.
(f) Use the formula you just wrote to calculate how much you need to invest each year for 10 years so that you will have $2,400,000 in 40 years.

Exercise 9 (Better late than never). On the other hand, you look at your finances now and decide that you couldn’t possibly invest now.
(a) Suppose you start investing in the same mutual fund in 10 years. Calculate how much you will have in 40 years if you invest $6,700 for 10 years, starting 10 years from now (so that the first $6,700 will only have 30 years to grow).
(b) Wow, that’s not nearly enough. Suppose that you keep investing $6,700 for the rest of the 30 years, starting in 10 years. How much will you have in 40 years then?
(c) That’s still not nearly enough. Calculate what you will need to invest every year for the rest of your working life, starting in 10 years, in order to have $2,400,000 in 40 years.
(d) Suppose you start investing in 20 years. Calculate what you will need to invest every year for the rest of your working life, starting in 20 years, in order to have $2,400,000 in 40 years.