Test 3
27–28 Oct 2005
Math 119, Section 1, Fall 2005
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No calculators, notes, or books.

Instructions: Read the questions carefully. Put your answers in the provided boxes. In order to receive full credit, you will need to neatly show your work on these pages and simplify your answers appropriately. Do not attach extra pages. If the provided space is insufficient, use the blank sides of adjacent pages.

1. (1 pt each) Compute the following integrals. Assume that \( n, k, \) and \( a \) are constants and \( a > 0 \) and \( a \neq 1 \) and \( n \neq -1 \).

\[
\begin{align*}
\int k \, dx &= kx + C \\
\int x^n \, dx &= \frac{x^{n+1}}{n+1} + C \\
\int kx \, dx &= \frac{kx^2}{2} + C \\
\int \frac{1}{x} \, dx &= \ln |x| + C
\end{align*}
\]

2. (7 pts) Find the area between the x-axis and \( f(x) = 6x^2 - 6x \) from \( x = 0 \) to \( x = 2 \).

First check where the function crosses the x-axis.

\[
\begin{align*}
6x^2 - 6x &= 0 \\
6x(x - 1) &= 0 \\
x &= 0, 1
\end{align*}
\]

\[
\begin{align*}
\int_0^2 (6x^2 - 6x) \, dx &= \left[ 2x^3 - 3x^2 \right]_0^2 \\
&= 2 - 3 + 16 - 12 - (2 - 3) \\
&= 1 + 5 = 6
\end{align*}
\]

3. (7 pts) Find the average value of the function \( f(x) = 3x^2 - 3 \) on the interval \([1, 3]\).

\[
\frac{1}{3-1} \int_1^3 3x^2 - 3 \, dx
\]

\[
= \frac{1}{2} \left( x^3 - 3x \right)
\]

\[
= \frac{1}{2} \left( (3)^3 - 3(3) \right) = \frac{1}{2} (27 - 9 - 1 - 3) = \frac{1}{2} (18 + 2) = 10
\]
4. (5 pts) Calculate \( \int 4m^3 + e^{2m} \, dm \).

\[
\frac{4m^4}{4} + e^{2m} \frac{2}{2} + C
\]

5. (5 pts) Calculate \( \int (1 - x^2)e^x \, dx \)

\[
\frac{D}{1 - x^2} + \frac{I}{e^x}
\]

\[
-2x + e^x
\]

Simplifying further gives

\[
(1 - x^2)e^x + 2xe^x - 2e^x + C = e^x(1 - x^2 + 2x - 2) + C
\]

\[
= -e^x(x^2 - 2x + 1) + C
\]

\[
\frac{(2x^3 + 1)^6}{60} + C
\]

6. (5 pts) Calculate \( \int x^2(2x^3 + 1)^9 \, dx \).

\[
U = 2x^3 + 1
\]

\[
du = 6x^2 \, dx
\]

\[
\frac{1}{6} \int u^9 \, du = \frac{1}{6} \frac{u^{10}}{10} + C = \frac{(2x^3 + 1)^{10}}{60} + C
\]

7. (5 pts) Calculate \( \int 3x^2 \ln x \, dx \)

\[
\frac{D}{\ln x} + \frac{I}{3x^2}
\]

\[
\frac{1}{x} \rightarrow x^3
\]

\[
= x^3 \ln x - \int \frac{1}{x} \cdot x^2 \, dx = x^3 \ln x - \int x^2 \, dx
\]

\[
= x^3 \ln x - \frac{x^3}{3} + C
\]
8. (6 pts) Calculate \( \int_{-4}^{-1} 10^t \, dt = \frac{10^t}{\ln 10} \bigg|_{-4}^{-1} \)
\[
= \frac{10^{-1}}{\ln 10} - \frac{10^{-4}}{\ln 10}
\]

9. (6 pts) Calculate \( \int_{x=1}^{x=5} \frac{\ln x}{x} \, dx = \int_{u=1}^{u=\ln 5} u \, du = \)
\[
\frac{(\ln 5)^2}{2} - \frac{1}{2} \ln 1 = 0.5 \ln 5
\]

10. (6 pts) Calculate \( \int_{x=0}^{x=1} \frac{x}{4 - x^2} \, dx = \int_{u=4}^{u=3} \frac{1}{u} \, du \)
\[
u = 4 - x^2
\]
\[
du = -2x \, dx \]
\[
\frac{1}{2} \, dx = x \, dx
\]
\[
=-\frac{1}{2} \ln 3 + \frac{1}{2} \ln 4
\]

You could further simplify this to
\[
\ln 3^{\frac{1}{2}} + \ln 4^{\frac{1}{2}} = \ln (3^{\frac{1}{2}} 4^{\frac{1}{2}}) = \ln \left( \frac{2}{\sqrt{3}} \right)
\]
11. (7 pts each) Determine whether the following integrals are convergent or divergent. If the integral is convergent, give the value of the integral. If the integral is divergent, put “divergent” as your answer. No credit will be given for an answer without the appropriate work.

(a) \( \int_2^\infty \frac{1}{\sqrt{x}} \, dx \).

\[
\lim_{b \to \infty} \left[ \int_2^b \frac{1}{\sqrt{x}} \, dx \right] = \lim_{b \to \infty} 2\sqrt{x} \bigg|_2^b \\
= \lim_{b \to \infty} 2\sqrt{b} - 2\sqrt{2} = \infty
\]

(b) \( \int_{-\infty}^0 3e^{4x} \, dx \).

\[
\lim_{b \to -\infty} \left[ \int_b^0 3e^{4x} \, dx \right] = \lim_{b \to -\infty} \frac{3}{4} e^{4x} \bigg|_b^0 \\
= \lim_{b \to -\infty} \frac{3}{4} e^0 - \frac{3}{4} e^{4b} = \frac{3}{4}
\]

12. (7 pts) Find the volume of the solid of revolution formed by rotating about the x-axis the region bounded by the curves \( y = \sqrt{x+1} \), \( y = 0 \), \( x = 0 \), and \( x = 3 \).

\[
\int_0^3 \pi (\sqrt{x+1})^2 \, dx \\
= \int_0^3 \pi x + \pi \, dx = \left[ \frac{\pi x^2}{2} + \pi x \right]_0^3 \\
= \frac{9\pi}{2} + 3\pi = \frac{15\pi}{2}
\]
13. (7 pts) The rate of change of a continuous flow of money is given by \( f(x) = 1000e^{-0.88x} \). Calculate the present value of the money flow at an interest rate of 12% compounded continuously for 5 years.

\[
\int_{0}^{5} 1000e^{-0.88x} e^{-12x} \, dx = \int_{0}^{5} 1000e^{-x} \, dx
\]

\[
= \left[ -1000e^{-x} \right]_{0}^{5} = -1000e^{-5} + 1000
\]

\[= 1000(1 - e^{-5})\]

14. (7 pts) The depth of water in a lake is changing at a rate of \( (3\sqrt{t}) \), where \( t \) is measured in days. On day 1, the depth of water is 100 meters. Write a formula for the depth of water in the lake for any given day \( t \).

Let \( h \) be the depth of water in the lake.

Then \( \frac{dh}{dt} = 3\sqrt{t} \)

so \( h = \int 3\sqrt{t} \, dt = \frac{2}{3}t^{3/2} + C \)

and \( 100 = 2\left(1^{3/2}\right) + C \Rightarrow C = 98 \), so \( h = 2t^{3/2} + 98 \)

15. (7 pts) Jason is traveling in a car and writes down the speeds in miles per hour at certain times, given in the table below. Estimate the distance that he traveled by using the trapezoid method to estimate the area below \( v(t) \) on the interval \([0, 6] \) using 3 subintervals.

<table>
<thead>
<tr>
<th>time ( t )</th>
<th>speed ( v(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>50</td>
</tr>
<tr>
<td>6</td>
<td>60</td>
</tr>
</tbody>
</table>

\[2 \left( \frac{1}{2}(40) + 30 + 50 + \frac{1}{2}(60) \right)\]

\[= 2(130) = 260 \text{ miles}\]