Second Order Analysis of Structures:  
Newton-Raphson Iteration  
vs.  
Constant Stiffness Iteration

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Abstract

In second order structural analysis, systems of nonlinear equations must be created and solved. This is commonly performed using the Newton-Raphson method to iteratively find an approximate solution to the system. Although the Newton-Raphson method requires a large amount of computation in each iteration (due to matrix triangularization), it converges very quickly. A variation of the Newton-Raphson method uses a constant stiffness matrix. This Constant Stiffness method requires less computation in each iteration, but converges much more slowly. Although both methods can be used in second order structural analysis, one is almost always better than the other for a given structure. The best method for a given structure is found by comparing the ratio of iterations performed by each method to the number of degrees of freedom in the structure divided by six. For large structures with many degrees of freedom, the Constant Stiffness method is often more efficient; for small structures with only a few degrees of freedom, the Newton-Raphson method has the advantage.

Introduction

In the field of structural engineering structures must be accurately analyzed in order to ensure public safety and economic efficiency in the design of new structures and the analysis of existing structures. The true analysis of structures is quite difficult and is usually approximated by linear or first-order analysis. This method creates and solves a system of linear equations and is sufficient for solving most structural analysis problems; however, first-order analysis makes the assumption that displacements are small in the structure after the load has been applied. This assumption is accurate in most structural analysis problems, but if displacements are expected to be large (such as in failure analysis), the first-order analysis is insufficient.

Second order analysis, on the other hand, takes into account any displacements of the structure, large or small. This analysis is performed by creating and iteratively solving a system of nonlinear equations. The Newton-Raphson method is commonly used to find an approximate solution to this system of nonlinear equations. An alternative method of solving nonlinear systems is referred to as the Constant Stiffness Method.

The purpose of this paper is to discover “Which of these two methods is better in the nonlinear analysis of a truss?” We will compare the advantages and disadvantages of each
method. In the first section, we discuss how nonlinear systems are used in structural analysis. The next two sections summarize how the Newton-Raphson method and the Constant Stiffness method work. The final section presents the results obtained in this research and the last section draws conclusions from the results.

**Nonlinear Systems in Structural Analysis**

In second order structural analysis, the following nonlinear equation must be solved: \(0 = Z(U) - F\) where \(Z\) is a nonlinear system of \(n\) equations. Each equation is a function of \(U\), a vector containing the displacements of each of \(n\) degrees of freedom in the structure. \(F\) is a force vector representing the loads applied to each degree of freedom in the structure. The goal is to find the value of \(U\), and therefore the total displacement experienced by each degree of freedom in the structure.

This system of equations is highly non-linear and is extremely difficult to solve directly. Instead, iterative methods are used to find an approximate solution, within a specified tolerance. Both the Newton-Raphson method and the Constant Stiffness method require the slope of the function. The slope of \(Z(U)\), \(\partial Z/\partial U\), is found by constructing a tangent stiffness matrix, \(K_T\), of size \(n \times n\). The development of \(K_T\) will not be discussed in this report. The nonlinear system can now be written as a linear system: \(\Delta U = K_T^{-1}(F - Z(U))\), where \(\Delta U\) is used to update \(U\) between iterations (Balling 5-21).

**Newton-Raphson Iteration**

The Newton-Raphson method begins with an initial guess for the value of \(U\). The function is evaluated at \(U\) and a new guess is generated based on the slope of the function \(Z(U)\), as shown previously. Figure 1 in the appendix shows an example of the Newton-Raphson method being used to find the root of \(f(x) = e^{(x-0.75)} - 1\). As can be seen in the figure, a new guess is found from the line tangent to the curve at the initial guess. At the new guess, the slope is recalculated and a third guess is found from the line tangent to the curve at the second guess. This process is repeated until the each new guess is very close to the previous one. Note how fast this method converges to the correct value. In three iterations, the approximate solution is very close to the actual solution.
Although the Newton-Raphson method converges quickly, an iteration in nonlinear structural analysis is not very efficient. Each iteration involves the triangularization of the tangent stiffness matrix. The algorithm used for this operation requires $n^3/6$ floating point operations (FLOPs), where $n$ is the size of the tangent stiffness matrix (Balling 4-67). The total number of FLOPs required for Newton-Raphson method can be calculated by the following equation:

$$\text{Total FLOPs} = i_n \times n^3/6$$  \hspace{1cm} \text{Equation 1}$$

(where $i_n$ is the number of iterations)

**Constant Stiffness Iteration**

The Constant Stiffness Method is very similar to the Newton-Raphson method, with only one major difference; the Constant Stiffness Method only creates and triangularizes the tangent stiffness matrix once, using the same one for each iteration. Figure 2 in the appendix illustrates the same principle as the Constant Stiffness Method being used to find the root of $f(x) = e^{x-0.75} - 1$. As can be seen, the same slope is used for each iteration. By using the same tangent stiffness matrix each time, the number of FLOPs for each iteration drops from $n^3/6$ to $n^2$ (the backwards substitution algorithm requires $n^2$ FLOPs).

Although the number of FLOPs drops significantly, the reduction in computing power required for each iteration comes with a high cost. Note how slowly this method converges compared to the Newton-Raphson method. After six iterations, the Constant Stiffness method did not even come close to approximating the correct answer. The total number of FLOPs required for Constant Stiffness method can be calculated by the following equation:

$$\text{Total FLOPs} = i_c \times n^2$$  \hspace{1cm} \text{Equation 2}$$

(where $i_c$ is the number of iterations)

**Results**

The difference between the two methods depends on both the size of the matrices involved (number of degrees of freedom of the structure) and the number of iterations required to meet the specified tolerance. If equation 1 and equation 2 are set equal to each other, we find the
point at which either method requires the same number of FLOPs. After combining these
equations and simplifying we come up with the following equation:

\[ \frac{i_c}{i_n} = \frac{n}{6} \]  

Equation 3

From equation 3 we can show that if \( \frac{i_c}{i_n} < \frac{n}{6} \), the Constant Stiffness required fewer FLOPs to
come to the same answer as the Newton-Raphson method, suggesting that the constant stiffness
method is more efficient for the given structure; conversely, we can show that if \( \frac{i_c}{i_n} > \frac{n}{6} \), the
Newton-Raphson method is more efficient.

In order to show how variations in the structure affect the number of iterations required
for each method, pseudocode used in the Computer Structural Analysis and Optimization course
(CE En 504) was coded in MATLAB. The MATLAB code is part of a program used to analyze
a plane truss.

The first two functions analyze a two-member plane truss with two degrees of freedom
\( n=2 \). Figure 3 in the appendix shows the truss. The first of these functions, \( nr(a,b) \), returns the
number of iterations required by the Newton-Raphson method to achieve a tolerance of 0.2%.
The second function, \( cs(a,b) \) returns the number of iterations required by the Constant Stiffness
method to achieve the same tolerance. The input variables, \( a \) and \( b \), are the truss dimensions
shown in figure 3.

To illustrate the differences between the methods \( nr(1,1) \) was run and returned a value of
3 iterations. When \( cs(1,1) \) was run, it returned a value of 13 iterations. After inserting these
values into the previous equation, we find that 13/3 > 2/6, suggesting that the Newton-Raphson
method would be more efficient. Another example shows that if we use values of \( a=5 \) and
\( b=1.55 \), \( nr(5,1.55) \) returns 78 iterations and \( cs(5,1.55) \) returns 153. Once again we find that
153/78 > 2/6, showing that the Newton-Raphson method would be more efficient.

In a second set of functions, \( nr1() \) & \( cs1() \), another truss was created with 16 degrees of
freedom \( n=16 \). This truss is shown in figure 4 in the appendix. These functions have no input
and when run, the results were as follows: \( nr1 = 2 \) and \( cs1 = 3 \). In this case 3/2 < 16/6 and the
Constant Stiffness method was more efficient. Even though the Constant Stiffness method
required more iterations, the increased number of degrees of freedom due to the size of the
required more computing power from the Newton-Raphson method than the Constant Stiffness
method.
Conclusion

The question of “Which method is better?” is very sensitive to the problem given. As was shown in the results section, the two member truss with only two degrees of freedom was consistently solved more efficiently by the Newton-Raphson method. The Newton-Raphson method was clearly more efficient for the two member truss because the triangularization was very easy with only two degrees of freedom. On the other hand, the twenty member truss with sixteen degrees of freedom illustrated an instance where the Constant Stiffness method solved the problem more efficiently.

If \( n \) is small, it is very likely that the Newton-Raphson method is more efficient. In structural analysis, however, it is not uncommon to analyze structures with tens of thousands of degrees of freedom. Another factor in choosing the correct solution is whether there is a large variation in slope in the function. If the slope does not vary a great deal, the Constant Stiffness method has the advantage; if the slope experiences drastic changes, the Newton-Raphson method is probably the best choice.

In order to be most efficient, a hybrid form of the two methods can be used. One way that this is done is to write the algorithm to create a stiffness matrix and use it for several iterations before creating a new one. In this way the algorithm converges much more quickly but is not constantly triangularizing a new stiffness matrix.
References

APPENDIX
**Figure 1.** Newton-Raphson Iteration of: $f(x) = e^{(x-0.75)} - 1$

**Figure 2.** Constant Stiffness Iteration of: $f(x) = e^{(x-0.75)} - 1$
Figure 3. Two member truss for analysis using nr(a,b) and cs(a,b)

Figure 4. Twenty member truss for analysis using nr1() and cs1()